On the q-convolution on the line

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Abstract

I continue the investigation of a q-analogue of the convolution on the line started in a joint work with Koornwinder and based on a formal definition due to Kempf and Majid. Two different ways of approximating functions by means of the convolution and convolution of delta functions are introduced. A new family of functions that forms an increasing chain of algebras depending on a parameter $s > 0$ is constructed. The value of the parameter for which the mentioned algebras are well behaved, commutative and unital is found. In particular a privileged algebra of functions belonging to the above family is shown to be the quotient of an algebra studied in the previous article modulo the kernel of a q-analogue of the Fourier transform. This result has an analytic interpretation in terms of analytic functions whose q-moments have a particular behaviour. The same result makes it possible to extend results on invertibility of the q-Fourier transform due to Koornwinder. A few results on invertibility of functions with respect to the q-convolution are also obtained and they are related to solving certain simple linear q-difference equations with polynomial coefficients.

1 Introduction

In [CK99] a q-analogue of the convolution on the line was defined (inspired by results in [KM94]), a few algebras under this new convolution were constructed and commutativity of these algebras was investigated.

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In particular, it was shown how commutativity of the q-convolution strongly relies on whether a function can be uniquely determined by its q-moments. The q-convolution defined therein had as a formal limit the usual convolution of C-valued functions on $\mathbf{R}$. The q-Fourier transform studied in [Koo97], involving E_q, intertwines the q-convolution product and ordinary product of functions.

The definition of the q-convolution was motivated by the results in [KM94], where Fourier transforms and convolution product were defined for braided covector algebras. However, [CK99] and the present paper are developed in a commutative setting and from an analytic point of view. On the other hand, since the braided line is commutative as an algebra, one could interpret many of our results as living in the braided setting. Remarks referring to the theory of braided groups and explaining results from a braided theoretical point of view appear in this paper wherever the author thought they could help to motivate definitions and results. They can easily be skipped by a reader who is only interested in classical q-analysis.

After recalling the basic definitions and the main results in [CK99], I prove properties of the q-convolution related to q-integrability, approximation of functions by means of the q-convolution, and existence of zero divisors in the family of algebras studied in [CK99].

I construct a new algebra of functions with respect to the q-convolution that will behave more like an algebra of distributions than like an algebra of functions. It consists of an increasing chain of algebras depending on a continuous parameter s. The elements will all be given by the q-Gaussian $e_{q^2}(-X^2)$ times an entire function f whose coefficients have a particular decreasing behaviour. In the terminology of [Ram92], where q corresponds to our q^{-1}, this corresponds to the fact that f is a function of q-exponential growth of order 1 and finite type (they type is related to s). Showing that on the algebras associated to a parameter $s < 1$ the q-moment problem is determined, I prove that those algebras are commutative and algebraic domains (i.e. they have no zero-divisors). They are also unital if $s \geq q^{\frac{1}{2}}$. In particular, the existence of a unit is a peculiar phenomenon of the q-case, showing that on this space the q-convolution may be seen as a mid-way between convolution of functions and convolution of distributions. In particular, the unit can be expressed in terms of Jackson's q-Bessel function $\mathcal{J}_{\frac{1}{2}}^{(1)}$ (see [KS98] and references therein). Existence of a unit element and existence of zero divisors were not treated extensively in [CK99], although they implicitly appear in the examples related to commutativity. It is easy to show that if an algebra under q-convolution has no zero divisors, then it will be commutative, but the converse is not always true.

I also show that the commutative, unital algebras corresponding to the value of the parameter s ranging in $[q^{\frac{1}{2}}, 1)$ must coincide. I provide a constructive method that associates to power series with a good behaviour, a function in this algebra having the given power series as generating series of its q-moments. This

will show that the algebra corresponding to $s \in [q^{\frac{1}{2}}, 1)$ is isomorphic to the the space of definition of a formal version of the q-Fourier transform quotiented by its kernel (as homomorphism). The formal q-Fourier transform coincides with Koornwinder's one on a particular subspace, namely, the ideal generated by the q-Gaussian $e_{q^2}(-x^2)$ with respect to the q-convolution. I use this results in order to construct a new operator inverting the formal q-Fourier transform, hence Koornwinder's q-Fourier transform. It coincides with the inversion operator described by Koornwinder in [Koo97] on their common domain of definition, but it is defined on a bigger space than the domain defined in [Koo97]. One can use this new inversion formula to obtain new relations between bases of various spaces. Other inversion formulas of the q-Fourier transform were obtained in the braided context and on different spaces in [KM94] and in [OR97].
Since the algebra I constructed is unital, the question of invertibility with respect to the q-convolution araises. Although the inverse does not necessarily belong to the algebra, I obtain a few results on invertibility in a somewhat extended algebra. I show how similarly to the classical case, inversion of a function with respect to the q-convolution, can be interpreted in terms of solving inhomogeneous q-difference equations with constant coefficients (i.e. particular q-difference equations with coefficients in $\mathbf{C}[x^{-1}]$). Those equations can be transformed into q-difference equation with polynomial coefficients, that are regular singular at 0 but not at infinity (see [Ram92] for the definition of regular singular), and whose characteristic equation has roots 1, q, q^2, ..., q^{n-1} if the equation has order n. Solutions of homogeneous q-difference equations of a class including the above class were already described by Adams in [Ada29] and methods for solving inhomogeneous equations are given in [Ada25]. See also [Ada31] for a survey on what was known on q-difference equations in the thirties. A solution of a q-differential equation with constant coefficients can therefore clearly be found without using the q-convolution, but in particular cases q-convolution can simplify the problem. Moreover, it can be used in order to determine the space of functions a solution can belong to and unicity of the solution in a given space of functions. In particular, if F is a function of the form $e_{q^2}(-x^2)$ times a function of q-exponential growth of order 1 and small enough finite type, I can give conditions on the q-differential operator with constant coefficients L under which the solution y of $Ly = F$ will be again of the same form as F.

2 Definitions and Notations

In this Section I recall the necessary background, the notation and the results of [CK99].
Let $q \in (0, 1)$ be fixed.
Denote as usual $(a;q)_k := \prod_{j=0}^{k-1}(1 - aq^j)$, $\quad (a;q)_\infty := \lim_{k\to\infty}(a;q)_k$,

$[k]_q := \frac{1-q^k}{1-q}$, $[k]_q! := \frac{(q;q)_k}{(1-q)^k}$, $\left[{k \atop j}\right]_q := \frac{[k]_q!}{[j]_q![k-j]_q!} = \frac{(q;q)_k}{(q;q)_j(q;q)_{k-j}}$.
For q-hypergeometric series the notation of Gasper & Rahman [GR90] will be followed.
In particular, we will need the functions

$$e_q(x) = \frac{1}{(x;q)_\infty} = \sum_{k=0}^{\infty} \frac{x^k}{(q;q)_k} \quad \text{and} \quad E_q(x) = \sum_{k=0}^{\infty} \frac{q^{\binom{k}{2}} x^k}{(q;q)_k} = (-x;q)_\infty \tag{2.1}$$

where the series expansion of $e_q(x)$ holds for $|x| < 1$.
The q-derivative of a function f at $x \neq 0$ is given by $(\partial f)(x) := \frac{f(x)-f(qx)}{(1-q)x}$, and the q-shift Q of f is given by $(Qf)(x) := f(qx)$. For $\gamma > 0$, $L(\gamma)$ denotes the q-lattice $\{\pm q^k \gamma \,|\, k \in \mathbf{Z}\}$.
For a function f on $L(\gamma)$ the q-integral over $L(\gamma)$ is denoted and defined by

$$\int_\gamma f = \int_{-\gamma\cdot\infty}^{\gamma\cdot\infty} f(t)\, d_q t := (1-q) \sum_{k=-\infty}^{\infty} \sum_{\epsilon=\pm 1} q^k \gamma\, f(\epsilon q^k \gamma), \tag{2.2}$$

provided the summation absolutely converges.
A function $f\colon x \mapsto f(x)$ may also be denoted as $f(X)$. This will be useful for functions like $fX^e\colon x \mapsto f(x)x^e$ and $e_{q^2}(-X^2)\colon x \mapsto e_{q^2}(-x^2)$.
For $\gamma > 0$, $\mathcal{I}_\gamma$ denotes the space of absolutely q-integrable functions on $L(\gamma)$, and $\mathcal{I}_\gamma^\infty$ denotes the subspace of functions $f \in \mathcal{I}_\gamma$ such that fX^e is in $\mathcal{I}_\gamma$ for every $e \in \mathbf{Z}_{\geq 0}$. For $f \in \mathcal{I}_\gamma^\infty$ define the *moments*, respectively *strict moments* of f by:

$$\mu_{e,\gamma}(f) := q^{\frac{e^2+e}{2}} \int_{-\gamma\cdot\infty}^{\gamma\cdot\infty} f(x)x^e\, d_q x, \quad \nu_{e,\gamma}(f) := q^{\frac{e^2+e}{2}} \int_{-\gamma\cdot\infty}^{\gamma\cdot\infty} |f(x)x^e|\, d_q x. \tag{2.3}$$

For a real number $\alpha > 0$, $\mathcal{I}_{\gamma,\alpha}^\omega$ denotes the space of functions *of left type* α on $L(\gamma)$ consisting of all $f \in \mathcal{I}_\gamma^\infty$ such that, for some $b > 0$, $|\mu_{e,\gamma}(f)| = O(q^{\frac{\alpha e^2}{2}} b^e)$ as $e \to \infty$ and $\mathcal{I}_{\gamma,\alpha}^{\mathrm{s}\omega}$ denotes the space of functions *of strict left type* α on $L(\gamma)$ consisting of all $f \in \mathcal{I}_\gamma^\infty$ such that, for some $b > 0$, $\nu_{e,\gamma}(f) = O(q^{\frac{\alpha e^2}{2}} b^e)$ as $e \to \infty$.
The space $\mathcal{I}_\gamma^\omega$ (resp. $\mathcal{I}_\gamma^{\mathrm{s}\omega}$) denotes the union of all $\mathcal{I}_{\gamma,\alpha}^\omega$ (resp. $\mathcal{I}_{\gamma,\alpha}^{\mathrm{s}\omega}$). The space $\mathcal{I}_{\gamma,>\alpha}^\omega$ (resp. $\mathcal{I}_{\gamma,>\alpha}^{\mathrm{s}\omega}$) denotes the union of all $\mathcal{I}_{\gamma,\beta}^\omega$ (resp. $\mathcal{I}_{\gamma,\beta}^{\mathrm{s}\omega}$) for $\beta > \alpha$.
By $\mathcal{H}^{\mathrm{D}}$ (resp $\mathcal{H}^{\mathrm{S}}$) I denote the space of all functions which are holomorphic on some disk (resp. strip) centered in 0 (resp. $\mathbf{R}$). Observe that if $f \in \mathcal{H}^{\mathrm{D}}$ then ∂f is defined also at $x = 0$. $\mathcal{E}$ will denote the space of entire analytic functions.

Every time I write $\mathcal{H}^{\mathrm{S}}$ or $\mathcal{H}^{\mathrm{D}}$ followed by one of the spaces denoted by $\mathcal{I}_\gamma$ with some index (i.e. $\mathcal{I}_\gamma$, $\mathcal{I}_\gamma^\infty$ or the spaces of strict left functions or left functions on $L(\gamma)$) I mean the space of functions contained in some sort of intersection. For instance, $\mathcal{H}^{\mathrm{D}}\mathcal{I}_\gamma^\infty$ will denote the space of the functions on $L(\gamma)$ belonging to $\mathcal{I}_\gamma^\infty$ which coincide within some disk centered in 0 with the restriction of a

(necessarily unique) holomorphic function on that disk, with the assumption that the $\{\pm q^k\gamma \,|\, k \in \mathbf{Z}_{\geq 0}\}$ is contained in the disk. This assumption is not restrictive since $\int_\gamma f = \int_{q^k\gamma} f$.

Note that a function in $\mathcal{H}^{\rm D}\mathcal{I}^\infty_\gamma$ cannot be uniquely determined by its restriction on a disk, so that the data of the values of the function on $L(\gamma)$ outside the disk should always be added. We recall that if f is of left type α or of strict left type α, then every polynomial times f is again so.

Definition 2.1 *Let $f \in \mathcal{I}^\infty_\gamma$ and let g be a function defined on some subset of $\mathbf{C}$. Then the q-*convolution product *$f *_\gamma g$ is the function given by*

$$(f *_\gamma g)(x) := \sum_{e=0}^{\infty} \frac{(-1)^e \mu_{e,\gamma}(f)}{[e]_q!} (\partial^e g)(x) \tag{2.4}$$

for $x \in \mathbf{C}$ such that the q-derivatives $(\partial^e g)(x)$ are well-defined for all $e \in \mathbf{Z}_{\geq 0}$ and the sum on the right converges absolutely.

For an $f \in \mathcal{I}^\infty_\gamma$ we will denote by $\mu_\gamma(f)$ and $\nu_\gamma(f)$ the series $\sum_{k=0}^\infty \frac{\mu_{k,\gamma}(f)t^k}{[k]_q!}$ and $\sum_{k=0}^\infty \frac{\nu_{k,\gamma}(f)t^k}{[k]_q!}$ respectively. In particualr, in [CK99] it was proved that for $f \in \mathcal{H}^{\rm D}\mathcal{I}^\omega_{\gamma'}$ and $g \in \mathcal{H}^{\rm D}\mathcal{I}^\omega_\gamma$

$$\mu_\gamma(f *_{\gamma'} g) = \mu_{\gamma'}(f)\mu_\gamma(g) = \mu_{\gamma'}(g *_\gamma f) \tag{2.5}$$

Let $\mathcal{I}^{\mathcal{E}}_\gamma$ and $\mathcal{I}^{s\mathcal{E}}_\gamma$denote the space of functions in $\mathcal{I}^\infty_\gamma$ for which $\mu_\gamma(f) \in \mathcal{E}$ and $\nu_\gamma(f) \in \mathcal{E}$ respectively. $\mathcal{H}^{\rm D}\mathcal{I}^{\mathcal{E}}_\gamma := \mathcal{H}^{\rm D}\mathcal{I}^\infty_\gamma \cap \mathcal{I}^{\mathcal{E}}_\gamma$ and $\mathcal{H}^{\rm S}\mathcal{I}^{\mathcal{E}}_\gamma := \mathcal{H}^{\rm D}\mathcal{I}^\infty_\gamma \cap \mathcal{I}^{\mathcal{E}}_\gamma$ and similarly, $\mathcal{H}^{\rm D}\mathcal{I}^{s\mathcal{E}}_\gamma := \mathcal{H}^{\rm D}\mathcal{I}^\infty_\gamma \cap \mathcal{I}^{s\mathcal{E}}_\gamma$ and $\mathcal{H}^{\rm S}\mathcal{I}^{s\mathcal{E}}_\gamma := \mathcal{H}^{\rm D}\mathcal{I}^\infty_\gamma \cap \mathcal{I}^{s\mathcal{E}}_\gamma$. One can check that formula (2.5) still holds for $f \in \mathcal{H}^{\rm D}\mathcal{I}^{\mathcal{E}}_{\gamma'}$ and $g \in \mathcal{H}^{\rm D}\mathcal{I}^{\mathcal{E}}_\gamma$.
In [CK99] the following results are achieved:

Proposition 2.2 *With the notation just introduced:*

1. *The class $\mathcal{H}^{\rm D}\mathcal{I}^\omega_\gamma$ is an algebra (not necessary unital) with respect to $*_\gamma$. Its subclass $\mathcal{H}^{\rm S}\mathcal{I}^\omega_\gamma$ is also an algebra (not necessary unital) and it is a left ideal of $\mathcal{H}^{\rm D}\mathcal{I}^\omega_\gamma$.*

2. *The class $\mathcal{H}^{\rm D}\mathcal{I}^{\rm s\omega}_\gamma$ is a subalgebra of $\mathcal{H}^{\rm D}\mathcal{I}^\omega_\gamma$. Its subclass $\mathcal{H}^{\rm S}\mathcal{I}^{\rm s\omega}_\gamma$ is a left ideal of $\mathcal{H}^{\rm S}\mathcal{I}^\omega_\gamma$, $\mathcal{H}^{\rm D}\mathcal{I}^\omega_\gamma$ and $\mathcal{H}^{\rm D}\mathcal{I}^{\rm s\omega}_\gamma$.*

3. *The classes $\mathcal{H}^{\rm S}\mathcal{I}^{\rm s\omega}_{\gamma,>c}$ (for $c \in [0,1)$) and $\mathcal{H}^{\rm S}\mathcal{I}^{\rm s\omega}_{\gamma,c}$ (for $c \in (0,1]$) are left ideals of $\mathcal{H}^{\rm S}\mathcal{I}^{\rm s\omega}_\gamma$ and of $\mathcal{H}^{\rm S}\mathcal{I}^\omega_\gamma$. Similar properties hold for $\mathcal{H}^{\rm D}\mathcal{I}^{\rm s\omega}_\gamma$.*

4. *Let $f, g \in \mathcal{H}^{\rm D}\mathcal{I}^\omega_\gamma$, $h \in \mathcal{H}^{\rm D}$. Then $(f *_\gamma g *_\gamma h)(x) = (g *_\gamma f *_\gamma h)(x)$ for every x where the product is defined. In particular, for every pair of ideals $I \subset J$ with $I,\, J \in \{\mathcal{H}^{\rm S}\mathcal{I}^{\rm s\omega}_\gamma,\, \mathcal{H}^{\rm S}\mathcal{I}^\omega_\gamma,\, \mathcal{H}^{\rm D}\mathcal{I}^\omega_\gamma,\, \mathcal{H}^{\rm D}\mathcal{I}^{\rm s\omega}_\gamma\}$, I is a left module over $J/[J,\, J]_*$, where $[J,\, J]_*$ denotes the commutator ideal.*

5. *A subalgebra A of $\mathcal{H}^{\rm D}\mathcal{I}_\gamma^\omega$ under q-convolution is commutative if every function in the commutator $[A,\, A]_*$ is determined by its q-moments. In particular, the q-moment problem is determined on $\mathcal{H}^{\rm S}\mathcal{I}^{\rm s\omega}_{\gamma,>\frac{1}{2}}$ so that $\mathcal{H}^{\rm S}\mathcal{I}^{\rm s\omega}_{\gamma,>c}$ is a commutative algebra for every $c \in [1/2, 1)$.* □

Statements *1*, *2* and *4* and the first part of statement *5* still hold if we replace $\mathcal{H}^{\rm D}\mathcal{I}_\gamma^\omega$, $\mathcal{H}^{\rm D}\mathcal{I}_\gamma^{\rm s\omega}$, $\mathcal{H}^{\rm S}\mathcal{I}_\gamma^\omega$ and $\mathcal{H}^{\rm S}\mathcal{I}_\gamma^{\rm s\omega}$ by $\mathcal{H}^{\rm D}\mathcal{I}_\gamma^{\mathcal{E}}$, $\mathcal{H}^{\rm D}\mathcal{I}_\gamma^{s\mathcal{E}}$ $\mathcal{H}^{\rm S}\mathcal{I}_\gamma^{\mathcal{E}}$ and $\mathcal{H}^{\rm S}\mathcal{I}_\gamma^{s\mathcal{E}}$ respectively. However, there is clearly no $\mathcal{E}$-counterpart for $\mathcal{H}^{\rm S}\mathcal{I}^{\rm s\omega}_{\gamma,>\frac{1}{2}}$.

Proposition 2.3 *Let $(\mathcal{F}_\gamma f)(y) := \int_{-\gamma.\infty}^{\gamma.\infty} E_q(-iqxy)\, f(x)\, d_qx$ and let $(\tilde{\mathcal{F}}_\gamma f)(y) := \sum_{k=0}^\infty \mu_{k,\gamma}(f)\, \frac{(-iy)^k}{(q;q)_k} = \mu_\gamma(f)(-iy(1-q)^{-1})$.*

1. *If $f \in \mathcal{I}_\gamma^\omega$ then $\tilde{\mathcal{F}}_\gamma f$ is well-defined and it is an entire analytic function. If moreover $f \in \mathcal{I}_\gamma^{\rm s\omega}$ then $\mathcal{F}_\gamma f$ is also well-defined and $\mathcal{F}_\gamma f = \tilde{\mathcal{F}}_\gamma f$.*

2. *Let $f \in \mathcal{H}^{\rm D}\mathcal{I}_\gamma^\omega$, $g \in \mathcal{H}^{\rm D}\mathcal{I}_{\gamma'}^\omega$. Then $f *_\gamma g \in \mathcal{H}^{\rm D}\mathcal{I}_{\gamma'}^\omega$ and $\tilde{\mathcal{F}}_\gamma(f *_\gamma g) = (\tilde{\mathcal{F}}_\gamma f)(\tilde{\mathcal{F}}_{\gamma'} g)$, so $\tilde{\mathcal{F}}_\gamma$ is an algebra homomorphism from the algebra $\mathcal{H}^{\rm D}\mathcal{I}_\gamma^\omega$ with convolution product to the algebra $\mathcal{E}$ with ordinary product. Its kernel is given by functions for which all q-moments are zero.*

3. *$\tilde{\mathcal{F}}_\gamma$ is an injective algebra homomorphism on all subspaces of $\mathcal{H}^{\rm D}\mathcal{I}_\gamma^\omega$ on which the q-moment problem is determined. In particular, it is injective on $\mathcal{H}^{\rm S}\mathcal{I}^{\rm s\omega}_{\gamma,>\frac{1}{2}}$, where it coincides with $\mathcal{F}_\gamma$.* □

Again, statements involving $\mathcal{I}_\gamma^\omega$ and $\mathcal{I}_\gamma^{\rm s\omega}$ still hold if we replace the upper index ω by $\mathcal{E}$ and $s\mathcal{E}$. This will allow us to extend the domain of the formal q-Fourier transform $\tilde{\mathcal{F}}_\gamma$ and its inverse. I will frequently make use of the following formulas, deduced from formulas (9.8) and (9.14) in [Koo97] (for $k \in \mathbf{Z}_{\geq 0}$):

$$\mu_{2k,\gamma}(e_{q^2}(-X^2)) = c_q(\gamma)\, q^{k^2-k}\, (q;q^2)_k \qquad \mu_{2k+1,\gamma}(e_{q^2}(-X^2)) = 0 \tag{2.6}$$

$$\mu_{2k,1}(E_{q^2}(-q^2X^2)) = b_q\, q^{2k^2+k}\, (q;q^2)_k \qquad \mu_{2k+1,1}(E_{q^2}(-q^2X^2)) = 0 \tag{2.7}$$

where $b_q = \int_{-1\cdot\infty}^{1\cdot\infty} E_{q^2}(-q^2x^2)\, d_qx$ and $c_q(\gamma) = \int_{-\gamma\cdot\infty}^{\gamma\cdot\infty} e_{q^2}(-x^2)\, d_qx$ were given [Koo97].

3 Properties of the convolution

In this Section I describe useful properties of the convolution that were not discussed in [CK99]. In particular I investigate the behaviour of the convolution product with respect to integration and approximation of functions.

Definition 3.1 *Let S be the linear map from $\mathcal{H}^{\rm D}$ to $\mathcal{E}$ mapping $\sum_r a_r x^r$ to $\sum_r (-1)^r q^{\binom{r}{2}} a_r x^r$. Let ε be the map from $\mathcal{H}^{\rm D}$ to $\mathbf{C}$ evaluating a function at 0.*

Remark 3.2 A motivation for the definition of S and ε comes from the braided Hopf algebra structure that the algebra of power series $\mathbf{C}[[x]]$ has (see [Maj95], [Maj93] and [Koo90]). The braiding isomorphism for this braided Hopf algebra is $\Psi(x^k \otimes x^l) := q^{kl} x^l \otimes x^k$ and the comultiplication is $\Delta(x^k) := \sum_{j=0}^{k} \begin{bmatrix} k \\ j \end{bmatrix}_q x^{k-j} \otimes x^j$. In this setting, the operators just defined are the counit ε given by $\varepsilon(x^k) := \delta_{k,0}$ and the braided antipode S mapping x to $-x$ and extended as a braided-antimultiplicative map. This braided Hopf algebra $\mathcal{A}$ is called the *braided line* and it is the simplest example of a braided covector algebra. ♠

Remark 3.3 The operator S followed by the map $f(x) \to f(-x)$ is Changgui Zhang's formal q-Borel transform (see [Zha99], where $q > 1$). In his article, Zhang used the q-Borel transform in order to define summability of certain divergent power series (the so-called q-Gevrey series originally introduced in [Béz93]) with a particular growth. ♠

Recall that Q is the q-shift operator such that $(Qf)(x) = f(qx)$.

Proposition 3.4 *Let $f \in \mathcal{I}^{s\mathcal{E}}_{\gamma,\alpha}$ and let $g \in \mathcal{H}^{\mathrm{D}}$. Then for every $p \in \mathbf{Z}$, $f\, Q^p S g \in \mathcal{I}^{\infty}_{\gamma}$. If f, g and $h \in \mathcal{H}^{\mathrm{S}}\mathcal{I}^{\mathrm{s}\omega}_{\gamma,>\frac{1}{2}}$ there holds*

$$\int_\gamma f\,(Q(Sg)) \;=\; \int_\gamma (Q(Sf))\,g \tag{3.1}$$

$$\int_\gamma (f *_\gamma g)\,Q(S(h)) \;=\; \int_\gamma (g *_\gamma h)\,Q(S(f)). \tag{3.2}$$

so in particular the restriction of $SQ = QS$ to $\mathcal{H}^{\mathrm{S}}\mathcal{I}^{\mathrm{s}\omega}_{\gamma,>\frac{1}{2}}$ is symmetric with respect to the q-integral.

Proof: Let $g \in \mathcal{H}^{\mathrm{D}}$ be such that $g(x) = \sum_{l=0}^{\infty} a_l x^l$ for $|x| < \rho$. Since $Sg \in \mathcal{E}$, for any fixed $k \in \mathbf{Z}_{\geq 0}$ and $p \in \mathbf{Z}$:

$$\int_\gamma |X^k (Q^p S g) f| \leq \int_\gamma \sum_{l=0}^{\infty} q^{\frac{l^2-l}{2}+lp} |a_l| |f X^{l+k}|$$
$$\leq C q^{\frac{1}{2}k^2 - kp} (\rho')^{-k} \sum_{l=0}^{\infty} ((\rho') q^{-\frac{1}{2}l} q^{-k+p})^{l+k} \nu_{l+k,\gamma}(f) < \infty$$

for some constants C and $\rho' > \rho$ and the first statement is proved.
Let now f and g in $\mathcal{H}^{\mathrm{S}}\mathcal{I}^{\mathrm{s}\omega}_{\gamma,>\frac{1}{2}}$. By statement 5 of Proposition 2.2 one has $\varepsilon(f *_\gamma g) =$

$\varepsilon(g *_\gamma f)$. By statement *2* in Proposition 2.2, $f *_\gamma g$ and $g *_\gamma f$ can be written as power series for $|x|$ sufficiently small. In particular

$$f *_\gamma g(x) = \sum_{r=0}^{\infty} \left[\sum_{k=0}^{\infty} (-1)^k \mu_{k,\gamma}(f) a_{r+k} \begin{bmatrix} r+k \\ k \end{bmatrix}_q \right] x^r$$

so that $\varepsilon(f *_\gamma g) = \sum_{k=0}^{\infty} (-1)^k q^{\frac{k^2+k}{2}} a_k \int_\gamma f X^k$. Since f is of strict left type we may interchange integration and summation by dominated convergence. Hence

$$\int_\gamma f\,(Q(Sg)) = \varepsilon(f *_\gamma g) = \varepsilon(g *_\gamma f) = \int_\gamma (Q(Sf))\, g \tag{3.3}$$

Formula (3.2) follows by (3.3) together with associativity of the q-convolution because

$$\int_\gamma (f *_\gamma g)\,(Q(Sh)) = \varepsilon((f *_\gamma g) *_\gamma h) = \varepsilon(f *_\gamma (g *_\gamma h)) = \int_\gamma (g *_\gamma h)\,(Q(Sf))$$

$\square$

Observe that formula (3.2) resembles the classical property that

$$\int_{-\infty}^{\infty} (f * g)(x) h(-x) dx = \int_{-\infty}^{\infty} (g * h)(x) f(-x) dx. \tag{3.4}$$

An important classical property of the convolution is that it makes it possible to approximate functions by sequences of functions with a nice behaviour. I introduce here a method to approximate functions by means of the q-analogue of the convolution.

Proposition 3.5 *Let $f \in \mathcal{I}_\gamma^{\mathcal{E}}$ be such that $\int_\gamma f = 1$. Define the sequence $f_k(x) := q^{-k} Q^{-k} f \in \mathcal{I}_\gamma^{\mathcal{E}}$ for $k \in \mathbf{Z}_{\geq 0}$. Let g be a function defined on a domain Ω together with all its q-derivatives. If for every $x \in \Omega$ there exists an $R_x > 0$ such that $|\partial^k g(x)| = O(R_x^k)$ as $k \to \infty$, then $\lim_{l \to \infty} f_l *_\gamma g(x) = g(x)$ pointwise for every $x \in \Omega$. If the majorization of $|\partial^k g(x)|$ is uniform, then the limit is uniform. In particular, this holds on a disk centered at zero if $g \in \mathcal{H}^{\mathrm{D}}$.*

Proof : Since $\mu_{l,\gamma}(f_k) = q^{kl} \mu_{l,\gamma}(f)$, for every fixed $x \in \Omega$, the product $f_l *_\gamma g$ at x is an absolutely convergent sum, so that

$$\Big| f_l *_\gamma g(x) - g(x) \Big| = \left| \sum_{r=1}^{\infty} \frac{(-1)^r q^{lr} \mu_{r,\gamma}(f)}{[r]_q!} \partial^r g(x) \right| \leq C q^l \sum_{r=1}^{\infty} \frac{|\mu_{r,\gamma}(f)| R_x^r}{[r]_q!}$$

for some constant $C > 0$. Hence for every fixed $x \in \Omega$, $|f_r *_\gamma g(x) - g(x)| \to 0$ as $l \to \infty$.
If for some $R > 0$ one has $|\partial^k g(x)| = O(R^k)$ as $k \to \infty$ for every $x \in \Omega$, then clearly $||f_l *_\gamma g - g||_\Omega \to 0$ as $l \to \infty$. $\square$

One would prefer to approximate functions by convolution from the right because $f *_\gamma g$ inherits the properties of g, not those of f. However one cannot have "good" convergence in this case in general, unless f has good properties too. Moreover, it has been shown in Theorem 6.5 and Example 6.7 in [CK99] that there are nonzero functions in $\mathcal{H}^{\mathrm{D}}\mathcal{I}_\gamma^{\mathrm{s}\omega}$ and in $\mathcal{H}^{\mathrm{D}}\mathcal{I}_{\gamma,\alpha}^{\omega}$ for every $\alpha > 0$ whose q-moments are all zero. Those functions could never be approximated using q-convolution from the right. In fact, the functions that could nontrivially be approximated using q-convolution from the right by a sequence of "good" functions are those for which the q-moment problem is determined, i.e. the functions that commute with "good" ones. In that case, approximation from the left and from the right coincide.

Example 3.6 Let $\gamma \in (0, 1)$ and $f_\gamma = \frac{e_{q^2}(-X^2)}{c_q(\gamma)}$, with $c_q(\gamma)$ is as in formula (2.6) and let $f_{k,\gamma} = q^{-k}Q^{-k}f_\gamma$ defined on $|\mathrm{Im}(x)| < q^k$. For any $g \in \mathcal{H}^{\mathrm{D}}$, $g(x) = \sum_l c_l x^l$ on a neighbourhood of 0 and the sequence of products:

$$(f_{k,\gamma} *_\gamma g)(x) = \sum_{l=0}^{\infty}(-i)^l c_l (q;q)_l \sum_{e=0}^{[\frac{l}{2}]} \frac{(-1)^e q^{e(e-1)} q^{2e(k+1)} (ix)^{l-2e}}{(q;q)_{l-2e}(q^2;q^2)_e} =$$

$$\sum_{l=0}^{\infty}(-i)^l q^{l(k+1)} c_l h_l(iq^{-(k+1)}x; q)$$

(where the $h_l(x;q)$'s are the discrete q-Hermite I polynomials, see [KS98]) converges to g uniformly on a disk centered at 0 for $k \to \infty$.
If $\gamma = 1$ and $F = \frac{E_{q^2}(-q^2X^2)}{b_q}$ with b_q as in formula (2.7) and if $F_k := q^{-k}Q^{-k}F$ for $k \geq 0$, the sequence of products

$$(F_k *_1 g)(x) = \sum_{l=0}^{\infty}(-i)^l q^{(k+l)l} c_l \tilde{h}_l(iq^{-(k+l)}x; q)$$

(where the $\tilde{h}_l(x;q)$'s are the discrete q-Hermite II polynomials, see [KS98]) converges uniformly to g on a disk for $k \to \infty$. ♠

We shall see now how the q-convolution product is related to the ordinary product of functions. One can prove that for two functions f and g,

$$\partial^n(fg) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q (Q^k \partial^{n-k} f)\, \partial^k g$$

(see also T. Koornwinder's informal note [Koo99]). Then, for f and g in $\mathcal{H}^{\mathrm{D}}$, and $h \in \mathcal{I}_\gamma^{\mathcal{E}}$, there holds

$$(h *_\gamma (fg)) = \sum_{e=0}^{\infty} \frac{(-1)^e \mu_{e,\gamma}(h)}{[e]_q!} \sum_{k=0}^{e} \begin{bmatrix} e \\ k \end{bmatrix}_q (Q^k \partial^{e-k} f)\, \partial^k g$$

$$= \sum_{k=0}^{\infty} \frac{(-1)^k q^{\frac{k^2+k}{2}} \partial^k g}{[k]_q!} \sum_{s=0}^{\infty} \frac{(-1)^s \mu_{s,\gamma}(X^k h)}{[s]_q!} \partial^s (Q^k f)$$

$$= \sum_{k=0}^{\infty} \frac{(-1)^k q^{\frac{k^2+k}{2}}}{[k]_q!} \Big(h\, X^k *_\gamma Q^k f \Big) \partial^k g.$$

In particular, for $g = X$ this implies

$$X(h *_\gamma f) = q(hX *_\gamma Qf) + h *_\gamma (fX) \tag{3.5}$$

i.e. multiplication by X obeys a sort of Leibniz rule.

The end of this Section is devoted to a few remarks about functions whose product is zero.

Lemma 3.7 *Let $f \in \mathcal{H}^{\mathrm{D}}\mathcal{I}^{\mathcal{E}}_\gamma$ and $g \in \mathcal{H}^{\mathrm{D}}\mathcal{I}^{\mathcal{E}}_{\gamma'}$. If $f *_\gamma g = 0$, then $\mu_\gamma(f) = 0$ and/or $\mu_{\gamma'}(g) = 0$. Hence, a convolution subalgebra A of $\mathcal{H}^{\mathrm{D}}\mathcal{I}^{\mathcal{E}}_\gamma$ has no zero divisors if and only if the q-moment problem is determined on A. In particular, $\mathcal{H}^{\mathrm{S}}\mathcal{I}^{\mathrm{s}\omega}_{\gamma,>\frac{1}{2}}$ is an algebraic domain.*

Proof: This is a trivial consequence of formula (2.5). The last statement follows by Lemma 6.1 in [CK99]. □

Corollary 3.8 *Let $f \in \mathcal{H}^{\mathrm{D}}\mathcal{I}^{\mathcal{E}}_\gamma$ and $g \in \mathcal{H}^{\mathrm{D}}\mathcal{I}^{\mathcal{E}}_{\gamma'}$. If $f *_\gamma g = 0$, then either $f *_\gamma h = 0$ for every h or $g *_{\gamma'} h = 0$ for every h for which the product is defined. In particular, either f or g is nilpotent.*

Proof: If $\mu_\gamma(f) \equiv 0$ then $f *_\gamma h = 0$ for every h. □

Remark 3.9 A consequence of the above results is that if a subalgebra $\mathcal{N}$ of $\mathcal{H}^{\mathrm{D}}\mathcal{I}^{\mathcal{E}}_\gamma$ has no zero divisors (i.e. if the q-moment problem is determined on $\mathcal{N}$), then $\mathcal{N}$ is commutative since by Corollary 5.11 in [CK99] $(f *_\gamma g - g *_\gamma f) *_\gamma h = 0$. The converse does not necessarily hold. Indeed, for $\gamma < 1$ consider the functions $e_{q^2}(-q^2X^2)$ and $e_q(iX)$. By formula (8.21) in [Koo97] for $t = q^{-1}$ one has $\mu_\gamma(e_q(iX)) \equiv 0$. Hence, $e_q(iX) *_\gamma f = 0$ for every $f \in \mathcal{H}^{\mathrm{D}}$. By formulas (2.6) $\mu_{2r+1,\gamma}(e_{q^2}(-q^2X^2)) = 0$ and $\mu_{2r,\gamma}(e_{q^2}(-q^2X^2)) = c_q(\gamma) q^{-2}(q;q^2)_r q^{r^2-r}$ (see also Section 9 in [Koo97]). Since $\partial^k e_q(ix) = i^k(1-q)^{-k} e_q(ix)$ one has

$$e_{q^2}(-X^2) *_\gamma e_q(iX) = q^{-2} c_q(\gamma) \sum_{r=0}^{\infty} \frac{(-1)^r q^{r^2-r}}{(q^2;q^2)_r} e_q(iX) = E_{q^2}(-1) e_q(iX) = 0 \tag{3.6}$$

Hence the subalgebra generated by $e_{q^2}(-q^2X^2)$ and $e_q(iX)$ is commutative although it has zero divisors. ♠

4 Discrete delta functions

This Section is devoted to the study of the q-convolution for discrete delta functions, and is based on ideas of T. Koornwinder.
On the q-lattice $L(\gamma)$ define the discrete delta functions

$$\delta_{\epsilon\gamma q^p}(\eta\gamma q^l) := \delta_{\epsilon,\eta}\delta_{l,p}$$

for any $\epsilon, \eta \in \{\pm 1\}$ and any l and $p \in \mathbf{Z}$. By Ryde's formula (see [Ryd21]) for iterated q-differentation at $x \neq 0$

$$(\partial^n f)(x) = (1-q)^{-n} x^{-n} \sum_{k=0}^{n} (-1)^k \begin{bmatrix} n \\ k \end{bmatrix}_q q^{-k(n-k)} q^{-\frac{k(k-1)}{2}} Q^k f(x)$$

and since $Q\delta_{\eta\gamma q^t} = \delta_{\eta\gamma q^{t-1}}$ one sees that $\partial^n \delta_{\eta\gamma q^t}$ is a linear combination of discrete δ functions $\delta_{\eta\gamma q^s}$ for $t - n \leq s \leq t$. In particular, for $t - n \leq s \leq t$ and $\epsilon \in \{\pm 1\}$,

$$(\partial^n \delta_{\eta\gamma q^t})(\epsilon\gamma q^s) = (-1)^{t-s} \delta_{\epsilon,\eta} \frac{\eta^n \gamma^{-n} q^{-sn}}{(1-q)^n} \begin{bmatrix} n \\ t-s \end{bmatrix}_q q^{-\frac{(t-s)(t-s-1)}{2} - (t-s)(n-t+s)}.$$

Moreover,

$$\mu_{k,\gamma}(\delta_{\eta\gamma q^n}) = (1-q)\eta^k q^{\frac{1}{2}(k^2+k)} \gamma^{k+1} q^{n(k+1)} \quad \text{and} \quad \nu_{k,\gamma}(\delta_{\eta\gamma q^n}) = |\mu_{k,\gamma}(\delta_{\eta\gamma q^n})|$$

hence the convolution product of discrete delta functions along $L(\gamma)$ is well-defined by Proposition 2.2. One computes, for $l \leq s$ and $\theta \in \{\pm 1\}$:

$$\begin{aligned}
&(\delta_{\epsilon\gamma q^t} *_\gamma \delta_{\eta\gamma q^s})(\theta\gamma q^l) = \sum_{k=0}^{\infty} \frac{(-1)^k q^{\frac{k^2+k}{2}} (1-q) \gamma^{k+1} \epsilon^k q^{t(k+1)}}{[k]_q!} \partial^k (\delta_{\eta\gamma q^s}(\theta\gamma q^l)) \\
&= \sum_{k=s-l}^{\infty} \frac{(-1)^k q^{\frac{k^2+k}{2}} (1-q) \gamma^{k+1} \epsilon^k q^{t(k+1)}}{[k]_q!} \times \\
&\times \frac{(-1)^{s-l} \delta_{\theta,\eta} \eta^k \gamma^{-k} q^{-lk}}{(1-q)^k} \begin{bmatrix} k \\ s-l \end{bmatrix}_q q^{-\frac{(s-l)(s-l-1)}{2} - (s-l)(k-s+l)} \\
&= \frac{\gamma(1-q)\delta_{\theta,\eta} q^{(t+s-l)+(s-l)(t-l)} (\eta\epsilon)^{s-l}}{(q;q)_{s-l}} (\eta\epsilon q^{(t-l+1)}; q)_\infty.
\end{aligned}$$

Hence, if $\eta = \epsilon$ one has

$$(\delta_{\eta\gamma q^t} *_\gamma \delta_{\eta\gamma q^s})(\theta\gamma q^l) = \frac{\gamma(1-q)\delta_{\theta,\eta} q^{(t+s-l)+(s-l)(t-l)}}{(q;q)_{s-l}} (q^{(t-l+1)}; q)_\infty$$

which is zero for $l > t$ so that the product is a linear combination of discrete delta functions with support $\eta\gamma q^l$ for $l \leq \min(s, t)$. For $l \leq t$ the product evaluated at

$\theta\gamma q^l$ is equal to

$$(\delta_{\eta\gamma q^t} *_\gamma \delta_{\eta\gamma q^s})(\theta\gamma q^l) = \frac{\gamma(1-q)\delta_{\theta,\eta}q^{(t+s-l)+(s-l)(t-l)}}{(q;q)_{s-l}(q;q)_{t-l}}(q;q)_\infty$$
$$= (\delta_{\eta\gamma q^s} *_\gamma \delta_{\eta\gamma q^t})(\theta\gamma q^l).$$

Hence two discrete delta functions commute if and only if their supports have the same signature. Therefore if two functions have support strictly contained in the same half line, we can formally show that they commute by writing the two functions as a sum of discrete delta functions. For instance, if f and g are functions defined on $L(\gamma)^+ := \{q^r\gamma \mid r \in \mathbf{Z}\}$ then we may write formally

$$f = \sum_{k=-\infty}^{\infty} f(q^k\gamma)\delta_{\gamma q^k} \quad \text{and} \quad g = \sum_{l=-\infty}^{\infty} g(q^l\gamma)\delta_{\gamma q^l}$$

so that their convolution product has also support in $L(\gamma)^+$ and formally:

$$(f *_\gamma g)(\gamma q^p) = \sum_{k=-\infty}^{\infty}\sum_{l=-\infty}^{\infty} f(q^k\gamma)g(q^l\gamma)(\delta_{q^k\gamma} *_\gamma \delta_{q^l\gamma})(q^p\gamma)$$
$$= \sum_{k=-\infty}^{\infty}\sum_{l=-\infty}^{\infty} f(q^k\gamma)g(q^l\gamma)(\delta_{q^l\gamma} *_\gamma \delta_{q^k\gamma})(q^p\gamma) = (g *_\gamma f)(\gamma q^p)$$
$$= (1-q)(q;q)_\infty(q^p\gamma)\sum_{h=0}^{\infty}\sum_{t=0}^{\infty}\frac{(Q^pf)(q^h\gamma)}{(q;q)_h}\frac{Q^pg)(q^t\gamma)}{(q;q)_l}q^{(h+t)+ht}.$$

Clearly f and g could not be analytic in a neighbourood of zero unless they are zero on $q^k\gamma$ for $k \geq k_0$. On the other hand, if $\eta \neq \epsilon$ the two products are different since the support of the product will be contained in the half line having the same signature as the discrete delta function on the right hand side of the product.

Remark 4.1 q-distributions and their q-Fourier transform have been studied in [OR97]. Olshanetski and Rogov define regular q-distributions as those distributions $D(\psi)$ for which there is a function ψ such that $D(\psi)(f) = \int_\gamma \overline{\psi} f$ for $\gamma = 1$. In particular, $D(\delta_{\eta q^k\gamma})(f) = (1-q)q^k\gamma f(\eta q^k\gamma)$. One can check in this case that

$$D(\partial\delta_{\eta q^k\gamma})(f) = -D(Q\delta_{\eta q^k\gamma})(\partial f) = -q^{-1}D(\delta_{\eta q^k\gamma})(Q^{-1}\partial f)$$

Observe that the (q-)regular distributions defined by $(1-q)^{-1}q^{-k}\delta_{\eta q^k\gamma}$ act as classical distributions on test functions vith real values, and their limit for $k \to \infty$ is the ordinary distribution given by Dirac's delta. ♠

5 The family $\mathcal{M}_s$

For any $s > 0$ let $\mathcal{M}_s$ be the family of functions of the form $F = f\, e_{q^2}(-X^2)$ where $f(x) = \sum_{l=0}^{\infty} a_l x^l$ with $|a_l| \leq C s^l q^{\frac{1}{2}l^2}$ for some $C > 0$.
It can be shown that if $s < q^{-\frac{1}{2}}$, such an f may also be written as $f(x) = \sum_{l=0}^{\infty} c_l \tilde{h}_l(x;q)$ with $|c_l| \leq C' s^l q^{\frac{l^2}{2}}$ for some $C' > 0$ *and same* s.
Here, $\tilde{h}_l(x;q) = (q;q)_k \sum_{l=0}^{[k/2]} \frac{(-1)^l q^{-2lk+2l^2+l} x^{k-2l}}{(q^2;q^2)_l\,(q;q)_{k-2l}}$ (see [KS98]) are the discrete q-Hermite II polynomials. Viceversa, if $f(x) = \sum_{l=0}^{\infty} c_l \tilde{h}_l(x;q)$ with $|c_l| \leq C' s^l q^{\frac{l^2}{2}}$ for some $C' > 0$ and some $s \in (0,\, q^{-\frac{1}{2}})$ then $f(x) = \sum_{l=0}^{\infty} a_l x^l$ with $|a_l| \leq C s^l q^{\frac{l^2}{2}}$ for some $C > 0$. This is achieved by means of formulas (8.9) and (8.17) in [Koo97], and the estimate in the proof of Theorem 6.5 in [CK99]. Clearly $\mathcal{M}_s \subset \mathcal{H}^{\mathrm{S}}$ for every $s > 0$ and $\mathcal{M}_s \subset \mathcal{M}_r$ if $s < r$.

Remark 5.1 In [Ram92], where $q > 1$, power series of the type above described are called q-Gevrey of order -1 and finite type. q-Gevrey series were first introduced in [Béz93], but only for positive type. It is shown in Proposition 2.1 in [Ram92] that the above conditions on the coefficients of a power series f imply that f has q-exponential growth of order 1 and finite type. In particular, For $F = f\, e_{q^2}(-X^2) \in \mathcal{M}_s$, f will have order -1 and finite type smaller or equal to $sq^{\frac{1}{2}}$. ♠

By a simple computation one sees that $\mathcal{M}_s \subset \mathcal{I}_\gamma^\infty$ for every $s > 0$ and for $\gamma > 0$. Indeed,

$$\int_\gamma |f\, e_{q^2}(-X^2) X^e| \leq C \sum_{n=0}^{\infty} s^n q^{\frac{n^2}{2}} \int_\gamma |X|^{n+e} e_{q^2}(-X^2)$$

$$\leq C' \sum_{n=0}^{\infty} q^{\frac{n^2}{2} - \frac{(n+e)^2}{2} + \frac{-n-e}{2} + \frac{1}{4}(n+e)^2} s^n b^{n+e} < \infty$$

for some positive constants C, C' and b. Here is used that $e_{q^2}(-X^2)$ is of strict left type 1/2, as it was shown in [CK99]. Hence, $\mathcal{M}_s \subset \mathcal{H}^{\mathrm{S}}\mathcal{I}_\gamma^\infty$ for every $\gamma \in (0,\, 1)$. However, the elements of $\mathcal{M}_s$ do not belong to $\mathcal{I}_\gamma^{\mathcal{E}}$ in general. Indeed, let $M(x) = e_{q^2}(-x^2) \sum_{n=0}^{\infty} q^{2n^2} s^n x^{2n}$ for some $s < 1$. One computes

$$\mu_{2e,\gamma}(M) = c_q(\gamma) q^e \sum_{n=0}^{\infty} q^{(n-e)^2} s^n (q;q^2)_{n+e}$$

where $c_q(\gamma)$ is as in formula (2.6). Since

$$\sum_{n=0}^{\infty} q^{(n-e)^2} s^{n-e+e} (q;q^2)_{n+e} \geq (q;q^2)_\infty s^e \sum_{n=e}^{\infty} q^{(n-e)^2} s^{n-e} = (q;q^2)_\infty s^e \sum_{p=0}^{\infty} q^{p^2} s^p$$

$|\mu_{2e,\gamma}(M)| \geq Cq^e s^e$ for some constant C hence $M \notin \mathcal{I}_\gamma^{\mathcal{E}}$.
If $g \in \mathcal{I}^{\mathcal{E}} - \gamma$ and $F \in \mathcal{M}_s$ for $s \in (0,\, q^{-\frac{1}{2}})$ it makes sense to compute $g *_\gamma F$, which will belong to $\mathcal{H}^{\mathrm{S}}\mathcal{I}_\gamma^\infty$ for every $\gamma < 1$ by Proposition 4.4 and Lemma 3.4 in [CK99].

Proposition 5.2 *For every $s \in (0,\, q^{-\frac{1}{2}})$, $\mathcal{M}_s$ is a left module for $\mathcal{H}^{\mathrm{D}}\mathcal{I}_\gamma^{\mathcal{E}}$, its subalgebras and their quotient by their respective commutator ideals.*

Proof: By (8.28) in [Koo97]

$$\partial^t(\tilde{h}_l(x;q)e_{q^2}(-x^2)) = \frac{(-1)^t q^{lt+\frac{t^2-t}{2}}}{(1-q)^t}\tilde{h}_{l+t}(x;q)e_{q^2}(-x^2). \tag{5.1}$$

Hence, for $g \in \mathcal{H}^{\mathrm{D}}\mathcal{I}_\gamma^{\mathcal{E}}$ and $F(x) = \sum_{t=0}^\infty a_t \tilde{h}_t(x;q)e_{q^2}(-x^2)$ one can compute

$$\begin{aligned} (g *_\gamma F)(x) &= \sum_{e=0}^\infty \frac{q^{\frac{e^2+e}{2}} \int_\gamma g\, X^e}{(q;q)_e} \sum_{p=e}^\infty a_{p-e} q^{(p-e)e+\frac{e^2-e}{2}} \tilde{h}_p(x;q)e_{q^2}(-x^2) && (5.2)\\ &= \sum_{p=0}^\infty \left[\sum_{e=0}^p \frac{q^{ep}\int_\gamma g\, X^e}{(q;q)_e} a_{p-e}\right] \tilde{h}_p(x;q)e_{q^2}(-x^2) && (5.3) \end{aligned}$$

where we could interchange summations by dominated convergence, using the estimate

$$\frac{q^{\frac{k^2-k}{2}}}{(q;q)_k}\,|\tilde{h}_k(x;q)| \leq \frac{(1-q)^{-1}\max(1,|x|)}{(q;q)_k} \sum_{p=0}^\infty \frac{q^{2p^2-p}|x|^{2p}}{(q^2;q^2)_p} \tag{5.4}$$

which was obtained in the proof of Theorem 6.5 in [CK99]. One has

$$\left|\sum_{e=0}^p \frac{q^{ep}\int_\gamma g(x)x^e}{(q;q)_e} a_{p-e}\right| \leq Bs^p q^{\frac{p^2}{2}} \sum_{e=0}^\infty \frac{|\mu_{e,\gamma}(g)|s^{-e}q^{-\frac{1}{2}e}}{(q;q)_e} = \tilde{D}q^{\frac{p^2}{2}}s^p$$

for nonnegative constants B and $\tilde{D}$. Hence $g *_\gamma \mathcal{M}_s \subset \mathcal{M}_s$. The fact that $(f *_\gamma g) *_\gamma F = f *_\gamma (g *_\gamma F)$ holds already for $g \in \mathcal{H}^{\mathrm{D}}$. Last statement follows by equation (2.5). □

In particular, for every $s \in (0,\, q^{-\frac{1}{2}})$ the familes $\mathcal{M}_s$ are left modules for all algebras of functions of left type, strict left type, etcetera.

Corollary 5.3 *For every $s \in (0, q^{-\frac{1}{2}})$ and for every γ the space $\mathcal{I}_\gamma^{\mathcal{E}} \cap \mathcal{M}_s$ is an algebra and $\mathcal{I}_\gamma^\omega \cap \mathcal{M}_s$ a subalgebra. For $s, r \in (0, q^{-\frac{1}{2}})$ with $s < r$, $\mathcal{I}_\gamma^{\mathcal{E}} \cap \mathcal{M}_s$ is an ideal of $\mathcal{I}_\gamma^{\mathcal{E}} \cap \mathcal{M}_r$ and a module over $\mathcal{H}^{\mathrm{S}}\mathcal{I}_\gamma^{\mathcal{E}} \cap \mathcal{M}_r / [\mathcal{H}^{\mathrm{S}}\mathcal{I}_\gamma^{\mathcal{E}} \cap \mathcal{M}_r,\, \mathcal{H}^{\mathrm{S}}\mathcal{I}_\gamma^{\mathcal{E}} \cap \mathcal{M}_r]_*$. Analogous statements hold when we replace everywhere the upper index $\mathcal{E}$ by ω.* □

It follows by the particular structure of the elements of $\mathcal{M}_s$ that for a fixed s, $f \in \mathcal{M}_s \cap \mathcal{I}^{\mathcal{E}}_{\gamma}$ if and only if $f \in \mathcal{M}_s \cap \mathcal{I}^{\mathcal{E}}_{\gamma'}$ for $\gamma, \gamma' < 1$ and that the two algebras are isomorphic because $f *_\gamma g = \frac{c_q(\gamma)}{c_q(\gamma')} f *'_\gamma g$. Therefore we could even remove the lower index γ from $\mathcal{I}_\gamma$, $*_\gamma$, μ_γ etcetera.

Remark 5.4 By formula (5.1) it follows by direct computation that every $\mathcal{M}_s$ with $s \in (0, q^{-\frac{1}{2}})$ is closed under q-differentiation.
Indeed if $F = (\sum_k c_k \tilde{h}_k(x;q))\, e_{q^2}(-X^2)$ with $|c_k| \le C s^k q^{\frac{1}{2}k^2}$, then $\partial^r F = (\sum_k d_k \tilde{h}_k(x;q))\, e_{q^2}(-X^2)$ with $|d_k| \le (C q^{-\frac{1}{2}r} s^{-r} (1-q)^{-r}) s^k q^{\frac{1}{2}k^2}$.
However, the family is not closed under ordinary multiplication by x or under q-shift Q. In general, there hold only the weaker formulas, for $s < q^{\frac{1}{2}}$: $X\mathcal{M}_s \subset \mathcal{M}_{sq^{-1}}$ and $Q\mathcal{M}_s \subset \mathcal{M}_{sq^{-1}}$ as it follows by direct computation, using $e_{q^2}(-x^2) = \frac{1}{(-x^2;q^2)_\infty}$. ♠

Example 5.5 Let $p(x) = \sum_{n=0}^{M} a_n x^n \in \mathbf{C}[x]$. Then $p(X)\, e_{q^2}(-X^2) \in \mathcal{M}_s$ for every s by taking $C = \max_n(|a_n|) q^{-\frac{M^2}{2}} s^{-M}$. Clearly $p(X)\, e_{q^2}(-X^2)$ is of strict left type $\frac{1}{2}$ since $e_{q^2}(-X^2)$ is (see Example 3.2 in [CK99]). ♠

Example 5.6 For $m \in \mathbf{Z}_{\ge 0}$ let

$$\begin{aligned} g_m(x) &:= e_{q^2}(-x^2)\, {}_0\phi_1(-; q^{1+2m}; q^2, -q^{1+2m}x^2) \\ &= e_{q^2}(-x^2) \sum_{r=0}^{\infty} \frac{(-1)^r q^{2r^2-r} q^{2mr} x^{2r}}{(q^{1+2m};q^2)_r (q^2;q^2)_r}\,. \end{aligned} \tag{5.5}$$

Those functions were constructed first in Example 6.7 in [CK99]. One sees immediately that $g_m \in \mathcal{M}_{q^{m-\frac{1}{2}}}$ for every $m \in \mathbf{Z}_{\ge 1}$. By the formula after (2.14) in [KS92], $g_m(x) = {}_2\phi_1(0,0;q^{2m+1}|q^2,-x^2)$ and the g_m's are related to Jackson's q-Bessel function $\mathcal{J}^{(1)}_\alpha(2x;q^2)$ by:

$$g_m(x) = \frac{(q^2;q^2)_\infty}{(q^{2m+1};q^2)_\infty} x^{-m+\frac{1}{2}} \mathcal{J}^{(1)}_{m-\frac{1}{2}}(2x;q^2) \tag{5.6}$$

(see [KS98], [KS92] and references therein). It was shown in [CK99] that for every $m \in \mathbf{Z}_{\ge 0}$, $g_m \in \mathcal{H}^{\mathrm{S}}\mathcal{I}^{\omega}_{\gamma,\alpha}$ for every $\alpha > 0$ because the g_m's are even and $\mu_{2k}(g_m) = 0$ for every $k \ge m$. The q-moments were explicitely computed: the odd ones are always zero and

$$\mu_{2k,\gamma}(g_m) = c_q(\gamma) q^{k^2+k} (q;q^2)_k \frac{(q^{2m-2k};q^2)_\infty}{(q^{1+2m};q^2)_\infty}$$

where $c_q(\gamma)$ is as usual. Hence convolution from the left by some g_m is equivalent to a genuine q-differential operator of order $2m$.

It was shown in [CK99] that g_0 and g_1 are not of strict left type. Using the three term recurrence relation for Jackson's q-Bessel function in Exercise 1.25 in [GR90], one sees that the same holds for all g_m's. In particular one could use the same three term recurrence relation in order to show that the g_m's are of left type for all α once this is shown for g_0 and g_1.
Clearly $u_\gamma := \frac{g_1}{\int_\gamma g_1}$ is a left unit for $\mathcal{I}^{\mathcal{E}}_\gamma \cap \mathcal{M}_s$ for every $s \in [q^{\frac{1}{2}}, q^{-\frac{1}{2}})$, hence for $\mathcal{H}^{\mathrm{S}}\mathcal{I}^{\omega}_\gamma \cap \mathcal{M}_s$ for s ranging in the same set. Later we will see that u_γ is also a right unit too, for $s \in [q^{\frac{1}{2}}, 1)$. Moreover, since by Lemma 4.3 in [CK99]

$$(\partial^k u_\gamma) *_\gamma f = \partial^k(u_\gamma *_\gamma f) = \partial^k f \qquad \forall k \in \mathbf{Z}_{\geq 0} \tag{5.7}$$

convolution by $\partial^k u_\gamma$ from the left coincides with applying ∂^k. ♠

For functions in $\mathcal{M}_s$ with $s < q^{-\frac{1}{2}}$ it is possible to improve Lemma 3.7.

Proposition 5.7 *Let $g \in \mathcal{I}^{\mathcal{E}}_\gamma$ and let $F \in \mathcal{M}_s$ with $s < q^{-\frac{1}{2}}$. Then $g *_\gamma F = 0$ iff $\mu_\gamma(g) = 0$ and/or $F = 0$. In particular, if g belongs to the annihilator of some nonzero function $F \in \mathcal{M}_s$ with $s < q^{-\frac{1}{2}}$, then $g *_\gamma f = 0$ for every f for which the product is defined. In particular the representation of $\mathcal{H}^{\mathrm{D}}\mathcal{I}^{\mathcal{E}}_\gamma/[\mathcal{H}^{\mathrm{D}}\mathcal{I}^{\mathcal{E}}_\gamma, \mathcal{H}^{\mathrm{D}}\mathcal{I}^{\mathcal{E}}_\gamma]_*$, of $\mathcal{H}^{\mathrm{D}}\mathcal{I}^{\omega}_\gamma/[\mathcal{H}^{\mathrm{D}}\mathcal{I}^{\omega}_\gamma, \mathcal{H}^{\mathrm{D}}\mathcal{I}^{\omega}_\gamma]_*$, and of $\mathcal{H}^{\mathrm{S}}\mathcal{I}^{\mathrm{s}\omega}_{\gamma,>\frac{1}{2}}$ on $\mathcal{M}_s$ are faithful.*

Proof: Use formula (5.3) and independence of q-Hermite polynomials in order to show that $g *_\gamma F = 0$ if and only if either $\mu_\gamma(g) = 0$ of $F = 0$. The last statement follows by Lemma 6.1 in [CK99]. □

Remark 5.8 Observe that the result in Proposition 5.7 does not necessary hold in general. Namely, the function with zero moments for a product equal to zero might not be the left one. Take for instance $e_{q^2}(-q^2X^2) *_\gamma e_q(iX) = 0$ that was computed in Remark 3.9. ♠

We will investigate commutativity. The following Lemma was communicated to me by T. Koornwinder.

Lemma 5.9 *Let $s \in (0, 1)$ and let $F = f\, e_{q^2}(-X^2) \in \mathcal{M}_s$ and even. Suppose also that for some $\gamma > 0$ we have: $\int_\gamma X^{2k} F = 0$ for all $k \in \mathbf{Z}_{\geq 0}$. Then $F = 0$.*

Proof: Let $F = f\, e_{q^2}(-X^2) \in \mathcal{M}_s$ with $f(x) = \sum_n a_{2n}x^{2n}$ and $|a_{2n}| \leq C\, s^n\, q^{2n^2}$ for some $C > 0$ and some $s \in (0, 1)$. It will be justified by dominated convergence that

$$\int_\gamma |f(x)|^2\, e_{q^2}(-x^2) = \lim_{m\to\infty} \int_\gamma \left(\sum_{k=0}^{m} \overline{a_{2k}}\, x^{2k}\right) f(x)\, e_{q^2}(-x^2)\, d_qx = 0.$$

Hence $f(\gamma q^k) = 0$ for all $k \in \mathbf{Z}$. So $f \in \mathcal{E}$ vanishes on a set with limit point 0. Hence $f = 0$ identically. For the proof of the dominated convergence note that, for $x > 0$,

$$\left(\sum_{n=0}^{\infty} |a_{2n}|\, x^{2n}\right)^2 \leq C \sum_{n=0}^{\infty} x^{2n} s^n \left(\sum_{k=0}^{n} q^{2k^2} q^{2(n-k)^2}\right)$$
$$\leq C' \sum_{n=0}^{\infty} x^{2n} s^n q^{n^2} \leq C'' E_{q^2}(qsx^2).$$

for constants C, C' and C''. Hence

$$\sum_{k=-\infty}^{\infty} |f(\gamma q^k)|^2\, e_{q^2}(-q^{2k}\gamma^2)\, q^k \leq D \sum_{k=-\infty}^{\infty} \frac{(-\gamma^2; q^2)_k\, q^k}{(-qs\gamma^2; q^2)_k} < \infty.$$

where D is a nonnegative constant. This completes the proof. $\square$

One can extend the above Lemma to prove an analogue result for all functions in $\mathcal{M}_s$ for $s < 1$.

Lemma 5.10 *Let $s \in (0,\, 1)$ and let $F = f\, e_{q^2}(-X^2) = \sum_{n=0}^{\infty} a_n x^n e_{q^2}(-x^2) \in \mathcal{M}_s$. If for some $\gamma > 0$ one has $\int_\gamma F\, X^k = 0$ for every $k \in \mathbf{Z}_{\geq 0}$ then $f = F = 0$.*

Proof: We write $f = f_0 + f_1$ where f_0 (resp. f_1) is the even (resp. odd) part of f. Then, $\int_\gamma F\, X^{2k} = \int_\gamma f_0\, e_{q^2}(-X^2) X^{2k} = 0$ for every $k \geq 0$ so, by Lemma 5.9 $f = f_1$. Then $\partial f \in \mathcal{M}_s$ by Remark 5.4 and it is even. $\int_\gamma \partial f = 0$ and $\int_\gamma X^k \partial f = -q^{-k}[k]_q \int_\gamma X^{k-1} f = 0$ for every $k \geq 1$. Apply Lemma 5.9 to get the statement. $\square$

In particular we have:

Corollary 5.11 *For every $\gamma > 0$ and for $s \in (0,\, 1)$, $\mathcal{I}_\gamma^{\mathcal{E}} \cap \mathcal{M}_s$ is a commutative algebra under the convolution product. If moreover, $s \in [q^{\frac{1}{2}},\, 1)$, $\mathcal{I}_\gamma^{\mathcal{E}} \cap \mathcal{M}_s$ is unital.*

Proof: Since by formula (2.5) $\mu_\gamma(f *_\gamma g - g *_\gamma f) = 0$, the first statement follows by Lemma 5.10. The second statement follows by commutativity, formula (5.7) for $k = 0$ and the fact that $u_\gamma \in \mathcal{I}_\gamma^{\omega} \cap \mathcal{M}_{q^{\frac{1}{2}}}$. $\square$

It follows that the subalgebras $\mathcal{I}_\gamma^{\omega} \cap \mathcal{M}_s$, $\mathcal{I}_\gamma^{s\mathcal{E}} \cap \mathcal{M}_s$ and $\mathcal{I}_\gamma^{s\omega} \cap \mathcal{M}_s$ are also commutative for $s \in (0,\, 1)$, and that $\mathcal{I}_\gamma^{\omega} \cap \mathcal{M}_s$ is unital for $s \in [q^{\frac{1}{2}},\, 1)$.

Corollary 5.12 *For every $\gamma > 0$ and for $s \in [q^{\frac{1}{2}},\, 1)$, $\mathcal{I}_\gamma^{\omega} \cap \mathcal{M}_s = \mathcal{I}_\gamma^{\omega} \cap \mathcal{M}_{q^{\frac{1}{2}}}$, and $\mathcal{I}_\gamma^{\mathcal{E}} \cap \mathcal{M}_s = \mathcal{I}_\gamma^{\mathcal{E}} \cap \mathcal{M}_{q^{\frac{1}{2}}}$.*

Proof: By commutativity and Corollary 5.3, $\mathcal{I}_\gamma^{\omega} \cap \mathcal{M}_{q^{\frac{1}{2}}}$ and $\mathcal{I}_\gamma^{\mathcal{E}} \cap \mathcal{M}_{q^{\frac{1}{2}}}$ are bilateral ideals of $\mathcal{I}_\gamma^{\omega} \cap \mathcal{M}_{q^{\frac{1}{2}}}$ and $\mathcal{I}_\gamma^{\mathcal{E}} \cap \mathcal{M}_{q^{\frac{1}{2}}}$ respectively, and they contain u_γ. Hence they coincide. $\square$

Example 5.13 Functions of the form $p(X)\, e_{q^2}(-X^2)$ where $p(X)$ is a polynomial function, form a commuting family of functions. They all belong to $\mathcal{I}_\gamma^{\mathrm{s}\omega}$ for every $0<\gamma<1$. However this class of functions is not closed under convolution product. Commutativity can also be checked directly as follows. Take $\tilde{h}_l(x;q)e_{q^2}(-x^2)$ for $l \in \mathbf{Z}_{\geq 0}$ as basis for the above space. By formulas (8.17) and (8.14) in [Koo97]

$$\int_\gamma e_{q^2}(-x^2)x^p\tilde{h}_r(x;q) = \begin{cases} c_q(\gamma)\frac{(q;q)_{r+2k}q^{-2rk-k^2-r^2}}{(q^2;q^2)_k} & \text{if } p-r=2k\geq 0; \\ 0 & \text{otherwise.} \end{cases}$$

Hence

$$\begin{aligned}&(\tilde{h}_r(X;q)e_{q^2}(-X^2) *_\gamma \tilde{h}_l(X;q)e_{q^2}(-X^2))(x) \\ &= c_q(\gamma)q^{lr}\sum_{k=0}^{\infty}\frac{q^{3k^2}q^{2(r+l)k}}{(q^2;q^2)_k}\tilde{h}_{l+r+2k}(x;q)e_{q^2}(-x^2)\end{aligned}$$

Since the last expression is symmetric in l and r commutativity holds, but the product will no longer be a polynomial times $e_{q^2}(-X^2)$. ♠

Example 5.14 The functions g_m for $m>0$ defined in Example 5.6 are a family of commuting functions in $\mathcal{M}_{q^{\frac{1}{2}}}$ by Lemma 5.9. On the contrary, $g_0 \notin \mathcal{M}_s$ for any $s<q^{-\frac{1}{2}}$ (its coefficients grow exactly like $q^{\frac{n^2}{2}}q^{-\frac{1}{2}n}$ as $n\to\infty$) and as it was shown in Example 6.7 in [CK99] it does not commute with g_1, hence with u_γ. On the other hand one can conclude that

$$g_m *_\gamma g_n = \sum_{r=0}^{m-1}\frac{\mu_{2r,\gamma}(g_m)}{(q;q)_{2r}}\partial^{2r}g_n = g_n *_\gamma g_m = \sum_{r=0}^{n-1}\frac{\mu_{2r,\gamma}(g_n)}{(q;q)_{2r}}\partial^{2r}g_m$$

for m, n in $\mathbf{Z}_{\geq 1}$. Using determinacy of the q-moment problem, one has for $m\in\mathbf{Z}_{\geq 1}$

$$g_m(x) = \frac{(q^{2m};q^2)_\infty}{(q^{1+2m};q^2)_\infty}\left(\sum_{k=0}^{\infty}\frac{(-1)^k q^{2mk}q^{2k^2-k}\tilde{h}_{2k}(x;q)}{(q^2;q^2)_k}\right)e_{q^2}(-X^2)$$

since both functions are in $\mathcal{M}_{q^{m-\frac{1}{2}}}\cap\mathcal{I}_\gamma^\omega$ and have the same q-moments. Hence we have the equality:

$${}_0\phi_1(-;q^{1+2m};q^2,-q^{2m+1}x^2) = \frac{(q^{2m};q^2)_\infty}{(q^{1+2m};q^2)_\infty}\left(\sum_{k=0}^{\infty}\frac{(-1)^k q^{2mk}q^{2k^2-k}\tilde{h}_{2k}(x;q)}{(q^2;q^2)_k}\right)$$

for $m\geq 1$. ♠

Corollary 5.15 *For every $s \in (0, q^{\frac{1}{2}}]$ and $r < q^{-\frac{1}{2}}$ the representation of $\mathcal{I}^{\mathcal{E}}_{\gamma} \cap \mathcal{M}_s$ on $\mathcal{M}_r$ is faithful. This implies, taking $r = s$ that $\mathcal{I}^{\mathcal{E}}_{\gamma} \cap \mathcal{M}_s$ and its subalgebras are algebraic domains.*

Proof: By Lemma 5.7 the annihilator of $\mathcal{M}_r$ in $\mathcal{I}^{\mathcal{E}}_{\gamma} \cap \mathcal{M}_s$ must be zero. $\square$

Corollary 5.16 *$\mathcal{H}^{\mathrm{D}}\mathcal{I}^{\omega}_{\gamma}/[\mathcal{H}^{\mathrm{D}}\mathcal{I}^{\omega}_{\gamma}, \mathcal{H}^{\mathrm{D}}\mathcal{I}^{\omega}_{\gamma}]_*$ is a unital algebra. The same holds if we replace the upper indices D by S and/or ω by $\mathcal{E}$. Moreover for the kernel of the formal q-Fourier transform $\tilde{F}_{\gamma}$ one has:*

$$Ker_{\mathcal{H}^{\mathrm{D}}\mathcal{I}^{\omega}_{\gamma}}(\tilde{\mathcal{F}}_{\gamma}) = \{f \in \mathcal{H}^{\mathrm{D}}\mathcal{I}^{\omega}_{\gamma} \mid \mu_{\gamma}(f) = 0\} = [\mathcal{H}^{\mathrm{D}}\mathcal{I}^{\omega}_{\gamma}, \mathcal{H}^{\mathrm{D}}\mathcal{I}^{\omega}_{\gamma}]_*$$
$$Ker_{\mathcal{H}^{\mathrm{S}}\mathcal{I}^{\omega}_{\gamma}}(\tilde{\mathcal{F}}_{\gamma}) = \{f \in \mathcal{H}^{\mathrm{S}}\mathcal{I}^{\omega}_{\gamma} \mid \mu_{\gamma}(f) = 0\} = [\mathcal{H}^{\mathrm{S}}\mathcal{I}^{\omega}_{\gamma}, \mathcal{H}^{\mathrm{S}}\mathcal{I}^{\omega}_{\gamma}]_*$$

and the same if we replace everywhere ω by $\mathcal{E}$.

Proof: The proof will be for $\mathcal{H}^{\mathrm{S}}\mathcal{I}^{\omega}_{\gamma}$, the other cases follow similarly. The function u_{γ} is a left unit in $\mathcal{H}^{\mathrm{S}}\mathcal{I}^{\omega}_{\gamma}$ hence its projection on the commutative algebra $\mathcal{H}^{\mathrm{S}}\mathcal{I}^{\omega}_{\gamma}/[\mathcal{H}^{\mathrm{S}}\mathcal{I}^{\omega}_{\gamma}, \mathcal{H}^{\mathrm{S}}\mathcal{I}^{\omega}_{\gamma}]_*$ is a unit therein. The first equality of the formula for the kernel in $\mathcal{H}^{\mathrm{S}}\mathcal{I}^{\omega}_{\gamma}$ is clear. Inclusion $\supseteq$ in the second equality follows by equation (2.5). Let $f \in \mathcal{H}^{\mathrm{S}}\mathcal{I}^{\omega}_{\gamma}$ be such that $\mu_{\gamma}(f) = 0$. Then $f = [u_{\gamma}, f]_*$, hence the other inclusion. $\square$

Observe that u_{γ} is not a left unit on $\mathcal{H}^{\mathrm{D}}\mathcal{I}^{\omega}_{\gamma}$ since $e_q(iX) *_{\gamma} u_{\gamma} = 0$.

6 The functions u_{γ} and $G_{k,\gamma}$ and topology

In this section we study of the functions g_m, defined in formula (5.5) and their q-derivatives, as they are of particular interest and useful in order to prove plenty of results.

Let $k, m \in \mathbf{Z}_{\geq 1}$ and let $g_{k,m} := \partial^k g_m$. By Remark 5.4, $g_{k,m} \in \mathcal{I}^{\omega}_{\gamma,\alpha} \cap \mathcal{M}_{q^{m-\frac{1}{2}}}$ for every α and for every $\gamma \in (0, 1)$. Moreover, by Lemma 4.1 in [CK99], we have

$$\mu_{l,\gamma}(g_{k,m}) = \begin{cases} 0 & \text{if } k + l \text{ odd}, \\ 0 & \text{if } l < k, \\ (-1)^k \frac{[l]_q!}{[k]_q!} \mu_{l-k}(g_m) & \text{otherwise.} \end{cases}$$

In particular for $k = 2p$ we have $\mu_{2j+1,\gamma}(\partial^{2p} g_m) = 0$ and $\mu_{2j,\gamma}(\partial^{2p} g_m) = 0$ for every $j \geq m + p$. One can see that $g_{2p,m}$ is not a multiple of g_{m+p} for $m \geq 1$ since $\mu_{2j,\gamma}(g_{m+2p}) \neq 0$ for $j < p$. On the other hand, $\partial^{2k} g_0 = \frac{(-1)^k}{(1-q)^{2k}} g_0$. This is checked using the fact that $g_0(x) = \frac{1}{2}(e_q(ix) + e_q(-ix)) = cos_q(x)$ (see Example 6.7 in [CK99]).

Let us introduce the family of functions in $\mathcal{M}_{q^{\frac{1}{2}}} \cap \mathcal{I}^{\omega}_{\gamma}$:

$$G_{k,\gamma} := \frac{(-1)^k g_{k,1}}{[k]_q! \int_{\gamma} g_1} = \frac{(-1)^k \partial^k u_{\gamma}}{[k]_q!} = \frac{(-1)^k (q^2; q^2)_{\infty} \partial^k \left(x^{-\frac{1}{2}} \mathcal{J}^{(1)}_{\frac{1}{2}}(2x; q^2) \right)}{(q^3; q^2)_{\infty} [k]_q!} \quad (6.1)$$

By direct computation one obtains:

$$G_{l,\gamma}(x)=\frac{e_{q^2}(-x^2)}{c_q(\gamma)(q;q)_l}\sum_{k=0}^{\infty}\frac{(-1)^k q^{\frac{1}{2}(2k+l)^2+\frac{1}{2}(2k-l)}\tilde{h}_{2k+l}(x;q)}{(q^2;q^2)_k} \tag{6.2}$$

and $G_{l,\gamma}$ differs from $G_{l,\gamma'}$ only by a multiplicity constant, since this is true for u_γ. Observe that $g_1(x)x(1-q)^{-1}=\sin_q(x)=\frac{1}{2i}(e_q(ix)-e_q(-ix))$ so that in particular the limit for $q\to 1^-$ of $g_1((1-q)x)x$ is $\sin(x)$.
The $G_{k,\gamma}$'s belong to $\mathcal{M}_{q^{\frac{1}{2}}}\cap\mathcal{I}^{\omega}_{\gamma,\alpha}$ for every α and every γ since $\mu_{r,\gamma}(G_{k,\gamma})=\delta_{r,k}$. $G_{k,\gamma}*_\gamma f=(-1)^k\frac{\partial^k}{[k]_q!}\partial^k f$ for every f for which the q-derivatives are defined, and

$$G_{k,\gamma}*_\gamma G_{l,\gamma}=\begin{bmatrix}k+l\\ l\end{bmatrix}_q G_{k+l,\gamma}. \tag{6.3}$$

The g_m's can be written as linear combinations of the $G_{r,\gamma}$'s using determinacy of the q-moment problem in $\mathcal{M}_{q^{\frac{1}{2}}}$. Indeed

$$g_m=\sum_{r=0}^{m}\mu_{2r}(g_m)\,G_{2r,\gamma}=\frac{c_q(\gamma)(q^{2m};q^2)_\infty}{(q^{1+2m};q^2)_\infty}\sum_{r=0}^{m}(-1)^r q^{2mr}(q;q^2)_r(q^{2-2m};q^2)_r G_{2r,\gamma}.$$

since both functions belong to $\mathcal{M}_{q^{\frac{1}{2}}}\cap\mathcal{I}^{\omega}_{\gamma}$ and have the same q-moments. This provides another way to express Jackson's q-Bessel functions using formula (5.6).
The expansion of functions in $\mathcal{M}_{q^{\frac{1}{2}}}\cap\mathcal{I}^{\omega}_{\gamma}$ in terms of the $G_{k,\gamma}$'s can be seen as an approximation of functions with respect to a suitable topology, i.e. the one determined by the multiplicity of a zero at $x=0$ of the q-moment series μ_γ. This is the subject of tha last part of this section.
Let $\gamma>0$ be fixed. For any $f\in\mathcal{I}^{\infty}_{\gamma}$ let $m(f)$ be the miminum nonnegative integer for which $\mu_{r,\gamma}(f)\neq 0$ (i.e. the multiplicity of a zero at $t=0$ of $\mu_\gamma(f)$ or the multiplicity of a zero at $y=0$ of $\tilde{\mathcal{F}}_\gamma(f)$).
Then put, for $f,\,g\in\mathcal{I}^{\infty}_{\gamma}$,

$$\mathrm{d}(f,\,g):=e^{-\mathrm{m}(f-g)}=\mathrm{d}(g,\,f) \tag{6.4}$$

so that for every f and g there holds $0\le\mathrm{d}(f,\,g)\le 1$. Let $f,\,g,\,h\in\mathcal{I}^{\mathcal{E}}_{\gamma}$. If $\mu_\gamma(f-h)=t^{\mathrm{m}(f-h)}\phi(t)$ and $\mu_\gamma(g-h)=t^{\mathrm{m}(g-h)}\psi(t)$ where ϕ and $\psi\in\mathcal{E}$ are such that $\psi(0)\phi(0)\neq 0$, then $\mu_\gamma(f-g)=t^{\min(\mathrm{m}(f-h),\,\mathrm{m}(g-h))}F(t)$ with $F\in\mathcal{E}$ and one has $\mathrm{m}(f-g)\ge\min(\mathrm{m}(f-h),\,\mathrm{m}(g-h))$. Hence

$$\mathrm{d}(f,\,g)\le e^{-\min(\mathrm{m}(f-h),\,\mathrm{m}(g-h))}=\max(e^{-\mathrm{m}(f-h)},\,e^{-\mathrm{m}(g-h)})\le\mathrm{d}(f,\,h)+\mathrm{d}(h,\,g) \tag{6.5}$$

Therefore d defines a metric on those subspaces of $\mathcal{I}^{\mathcal{E}}_{\gamma}$ on which the q-moment problem is determined. For instance, d is a metric on the spaces $\mathcal{M}_s\cap\mathcal{I}^{\mathcal{E}}_{\gamma}$ with $s\in(0,\,q^{\frac{1}{2}}]$, the algebras $\mathcal{H}^{\mathrm{S}}\mathcal{I}^{\mathrm{s}\omega}_{\gamma,>\frac{1}{2}}$, $\mathcal{H}^{\mathrm{D}}\mathcal{I}^{\omega}_{\gamma}/[\mathcal{H}^{\mathrm{D}}\mathcal{I}^{\omega}_{\gamma},\,\mathcal{H}^{\mathrm{D}}\mathcal{I}^{\omega}_{\gamma}]_*$ and $\mathcal{H}^{\mathrm{D}}\mathcal{I}^{\mathrm{s}\omega}_{\gamma}/\mathrm{Ann}(u_\gamma)$,

where Ann denotes the annihilator ideal.
Let f, g and h belong to an algebra with respect to the q-convolution product. Then by formula (2.5) one has $\mathrm{m}(f *_\gamma g) = \mathrm{m}(f) + \mathrm{m}(g)$ so that $\mathrm{d}(f *_\gamma g,\, h *_\gamma g) = \mathrm{d}(f,\, h)\mathrm{d}(g,\, 0) \leq \mathrm{d}(f,\, h)$. Hence the q-convolution product with g for a fixed g is continuous with respect to this topology. In particular, convolution by $\partial^k u_\gamma$, i.e. q-differentation, is continuous, as can be seen by the fact that $\mathrm{m}(\partial^k f) = \mathrm{m}(f) + k$. We have:

Lemma 6.1 *Let $g \in \mathcal{I}_\gamma^\infty$ be such that $\mu_\gamma(g)$ converges absolutely at least on the closed disk centered at zero and with radius $(1-q)^{-1}$. Then the sequence of functions $F_n(g) := \sum_{k=0}^n \mu_{k,\gamma}(g) G_{k,\gamma}$ converges to a well-defined function F. If the radius of convergence of $\mu_\gamma(g)$ is strictly greater than $(1-q)^{-1}q^{-1}$ or if $\gamma < 1$ and $\rho < (1-q)^{-1}2\gamma^{-1}$ then $F \in \mathcal{I}_\gamma^\infty$ and $\mu_\gamma(F) = \mu_\gamma(g)$.*

Proof: Define $F_n := \sum_{k=0}^n \mu_{k,\gamma}(g) G_{k,\gamma}$ for every $n \in \mathbf{Z}_{\geq 0}$. By formula (6.2) and the estimate (5.4) we have

$$\begin{aligned} &\left|\mu_{k,\gamma}(g)\, G_{k,\gamma}(x)\right| \\ &\leq C \max(1,\, |x|)\, |e_{q^2}(-x^2)|\, |\mu_{k,\gamma}(g)| \left[\sum_{p=0}^\infty \frac{q^{2p^2-p}|x^{2p}|}{(q^2;q^2)_p}\right] \left(\sum_{l=0}^\infty \frac{q^{2l}}{(q^2;q^2)_l}\right) \end{aligned}$$

for some constant C. Hence F_n tends to a well-defined function F as $n \to \infty$ if $\sum_{k=0}^\infty |\mu_{k,\gamma}(g)| < \infty$.
Next we want to prove that if the radius of convergence ρ of $\mu_\gamma(g)$ is stricly greater than $(1-q)^{-1}q^{-1}$ or if $\gamma < 1$ and $\rho > 2(1-q)^{-1}\gamma^{-1}$ then $F \in \mathcal{I}_\gamma^\infty$. If $\rho > (1-q)^{-1}q^{-1}$ one uses the fact that $u_\gamma = \sum_{k=0}^\infty c_k x^k e_{q^2}(-x^2)$ with $|c_k| \leq C_0 q^{\frac{1}{2}(k^2+k)}$ for some constant C_0 together with Remark 5.4 in order to conclude that for every $l \in \mathbf{Z}_{\geq 0}$

$$\int_\gamma |G_{p,\gamma}|\, |X^l| \leq C_0 \frac{q^{-p}}{(q;q)_p} \sum_{k=0}^\infty q^{\frac{1}{2}(k^2+k)} q^{-\frac{1}{2}(l+k)^2 - \frac{1}{2}(l+k)} \nu_{l+k,\gamma}(e_{q^2}(-X^2))$$

where $\nu_{k,\gamma}$ is defined in formula (2.3). Since it can be shown that $\nu_{r,\gamma}(e_{q^2}(-X^2)) \leq 2q^{-\frac{1}{4}} c_q(\gamma) q^{\frac{r^2}{4}}$ one obtains:

$$\int_\gamma |F X^l| \leq C_1 q^{-\frac{1}{2}(l^2+l)} \sum_{p=0}^\infty \frac{q^{-p}|\mu_{p,\gamma}(g)|}{(q;q)_p} \sum_{k=0}^\infty q^{\frac{(l-k)^2}{4}} < \infty$$

hence $F \in \mathcal{I}_\gamma^\infty$ and it is clear then that $\mu_\gamma(F) = \mu_\gamma(g)$. If $\rho > 2(1-q)^{-1}\gamma^{-1}$ and $\gamma < 1$ we may use Lemma 3.5 in [CK99] in order to show that for every $l \in \mathbf{Z}_{\geq 0}$

$$\int_\gamma |G_{p,\gamma}|\, |X^l| \leq \frac{1}{[p]_q!} \Big(\int_\gamma |X^l||u_\gamma| + r^l\, B \Big) \frac{2^p}{\gamma^p (1-q)^p}$$

for some constants $B > 0$ and $r \in (\gamma, 1)$. Then the proof follows as in the previous case. $\square$

Observe that it follows by the proof of the above Lemma that if g is any function of $\mathcal{I}_\gamma^\infty$ for which $\mu_\gamma(g)$ has a big enough radius of convergence, then $g *_\gamma u_\gamma$ is well defined.

Lemma 6.2 *Let $g \in \mathcal{I}_\gamma^\infty$ be such that $\mu_\gamma(g)$ converges absolutely at least on the closed disk centered at zero and with radius $(1-q)^{-1}$. Then the function $F := \sum_{k=0}^\infty \mu_{k,\gamma}(g) G_{k,\gamma}$ belongs to $\mathcal{M}_s$ for some $s < q^{-\frac{1}{2}}$. If the radius of convergence ρ of $\mu_\gamma(g)$ is strictly greater than $(1-q)^{-1}q^{-1}$ then $F \in \mathcal{M}_{q^{\frac{1}{2}}}$.*

Proof: By dominated convergence

$$F(x) = \frac{e_{q^2}(-x^2)}{c_q(\gamma)} \sum_{p=0}^\infty q^{\frac{1}{2}(p^2-p)} \tilde{h}_p(x;q) \sum_{k=0}^{[\frac{p}{2}]} \frac{\mu_{p-2k,\gamma}(g) q^{2k}}{(q;q)_{p-2k}(q^2;q^2)_k}$$

Therefore the coefficients c_p of the expansion of $F\, E_{q^2}(X^2)$ with respect to the discrete q-Hermite II polynomials are majorized by $q^{\frac{1}{2}(p^2-p)} \sum_{k=0}^{[\frac{p}{2}]} |\mu_{p-2k,\gamma}(g)| q^{2k}$ times some constant. If $\rho > (1-q)^{-1}$ then $|\mu_{l,\gamma}(g)| = O(a^l)$ for $l \to \infty$ for some $a \in (0,1)$. One can always assume that $a \in (q, 1)$. Then $|c_p| \le C q^{\frac{1}{2}p^2} (q^{-\frac{1}{2}} a)^p$ for some constant C. If $\rho > (1-q)^{-1}q^{-1}$ then a can be chosen in $(0, q)$. In that case $|c_p| \le C' q^{\frac{1}{2}(p^2-p)} q^p$ for some constant C'. $\square$

Corollary 6.3 *Let $g \in \mathcal{M}_s$ with $s < 1$ be such that $\mu_\gamma(g)$ converges absolutely at least on the closed disk centered at zero and with radius $(1-q)^{-1}q^{-1}$. Then g can be approximeted by finite linear combinations of the $G_{k,\gamma}$'s.*

Proof: By the above results and Lemma 5.10 there follows that $F = g$. $\square$

We have just seen that we can approximate various classes of functions by means of u_γ and its q-derivatives, as in classical distribution theory one approximates generalized functions by the delta functions and its derivatives.

Moreover, the methods used in the proof of Lemma 6.1 and Lemma 6.2 show that, if $f \in \mathcal{I}_\gamma^\infty$ is such that $\mu_\gamma(f)$ has a good behaviour, there exists a function $F \in \mathcal{I}_\gamma^{\mathcal{E}} \cap \mathcal{M}_{q^{\frac{1}{2}}}$ such that $\mu_\gamma(f) = \mu_\gamma(F)$. This is quite an interesting result because F belongs to a space that does not depend essentially on γ. Moreover, Lemma's 6.1 and 6.2 provide a *constructive* way to associate to a n entire q-moment series a unique function in $\mathcal{M}_{q^{\frac{1}{2}}} \cap \mathcal{I}_\gamma^{\mathcal{E}}$. This proves the following theorem

Theorem 6.4 *The elements of $\mathcal{M}_{q^{\frac{1}{2}}} \cap \mathcal{I}_\gamma^{\mathcal{E}}$ form a set of representatives of the quotients $\mathcal{H}^{\mathrm{D}}\mathcal{I}_\gamma^{\mathcal{E}} / [\mathcal{H}^{\mathrm{D}}\mathcal{I}_\gamma^{\mathcal{E}}, \mathcal{H}^{\mathrm{D}}\mathcal{I}_\gamma^{\mathcal{E}}]_*$ and $\mathcal{H}^{\mathrm{S}}\mathcal{I}_\gamma^{\mathcal{E}} / [\mathcal{H}^{\mathrm{S}}\mathcal{I}_\gamma^{\mathcal{E}}, \mathcal{H}^{\mathrm{S}}\mathcal{I}_\gamma^{\mathcal{E}}]_*$. Those algebras are all isomorphic. The same result holds if we replace everywhere the upper index $\mathcal{E}$ by ω. The projection modulo the commutator ideal is given in all cases by $f \mapsto f *_\gamma u_\gamma$.*

Proof: The bijection is clear by the discussion above. The fact that it is an algebra isomorphism follows from the fact that in those algebras the product is determined by μ_γ. $\square$

Observe that even though $\mathcal{H}^{\mathrm{S}}\mathcal{I}^{\omega}_{\gamma}$ is strictly contained in $\mathcal{H}^{\mathrm{D}}\mathcal{I}^{\omega}_{\gamma}$ and $[\mathcal{H}^{\mathrm{S}}\mathcal{I}^{\omega}_{\gamma},\, \mathcal{H}^{\mathrm{S}}\mathcal{I}^{\omega}_{\gamma}]_*$ is strictly contained in $[\mathcal{H}^{\mathrm{D}}\mathcal{I}^{\omega}_{\gamma},\, \mathcal{H}^{\mathrm{D}}\mathcal{I}^{\omega}_{\gamma}]_*$ (the function $\delta_{q^{-1}\gamma}$ belongs to $\mathcal{H}^{\mathrm{D}}\mathcal{I}^{\omega}_{\gamma}$ but not to $\mathcal{H}^{\mathrm{S}}\mathcal{I}^{\omega}_{\gamma}$), Theorem 6.4 states that for every function f in $\mathcal{H}^{\mathrm{D}}\mathcal{I}^{\omega}_{\gamma}$ there is always a function $f' \in \mathcal{H}^{\mathrm{S}}\mathcal{I}^{\omega}_{\gamma}$ such that $\mu_\gamma(f - f') \equiv 0$.

Observe also that the above result together with Corollary 5.16 imply that the kernel of $\tilde{F}_\gamma$ on $\mathcal{H}^{\mathrm{D}}\mathcal{I}^{\mathcal{E}}_{\gamma}$ does not depend on γ essentially.

Remark 6.5 On $\mathcal{I}^{\mathcal{E}}_{\gamma} \cap \mathcal{M}_{q^{\frac{1}{2}}}$ and $\mathcal{I}^{\omega}_{\gamma} \cap \mathcal{M}_{q^{\frac{1}{2}}}$ one can define the operators $\tilde{X}$ and $\tilde{Q}$ as: $\tilde{X}.f = (Xf) *_\gamma u_\gamma$ and $\tilde{Q}.f = (Qf) *_\gamma u_\gamma$. It follows by formula (3.5) that $\tilde{X}$ acts as a q-derivation. In particular, $\tilde{X}.G_{k,\gamma} = q^{-k} G_{k-1,\gamma}$ for $k \geq 1$ and $\tilde{X}.u_\gamma = 0$ as for the classical delta function (the unit with respect to the convolution). As for the q-shift operator: $\tilde{Q}.G_{k,\gamma} = q^{-k-1} G_{k,\gamma}$. Hence $\tilde{Q}\tilde{X} = q\tilde{X}\tilde{Q}$.

Lemma's 6.1 and 6.2 can be generalised.

Lemma 6.6 *Let $h \in \mathcal{M}_s$ for $s < q^{-\frac{1}{2}}$ and let $g \in \mathcal{I}^{\infty}_{\gamma}$ be such that $\mu_\gamma(g)$ converges absolutely at least on the closed disk centered at zero and with radius $(1-q)^{-1} q^{-\frac{1}{2}} s^{-1}$. Then $g *_\gamma h$ is a well-defined function in $\mathcal{M}_s$.*

Proof: This Lemma generalises the result of Lemma 6.2 where $h = u_\gamma$. One proves it similarly writing h as $e_{q^2}(-X^2)\, f$, expanding f with respect to the discrete q-Hermite II polynomials, and using the majorization of the coefficients of the expansion of $\partial^k h$ given in Remark 5.4. $\square$

Let $s < q^{-\frac{1}{2}}$. By $\mathcal{I}^{\rho,s}_{\gamma}$ we denote the space of functions in $\mathcal{I}^{\infty}_{\gamma}$ for which $\mu_\gamma(f) \in \mathcal{H}^{\mathrm{D}}$ and has a radius of convergence greater than $(1-q)^{-1} q^{-\frac{1}{2}} s^{-1}$.

Corollary 6.7 *For $s < q^{-\frac{1}{2}}$, $\mathcal{M}_s \cap \mathcal{I}^{\rho,s}_{\gamma}$ is an algebra and equation (2.5) holds for functions in $\mathcal{M}_s \cap \mathcal{I}^{\rho,s}_{\gamma}$. If $s < 1$ the algebra is commutative. If $s = q^{\frac{1}{2}}$, $\mathcal{M}_{q^{\frac{1}{2}}} \cap \mathcal{I}^{\rho,q^{\frac{1}{2}}}_{\gamma}$ is unital and all $\mathcal{M}_s \cap \mathcal{I}^{\rho,s}_{\gamma}$ coincide for $s \in [q^{\frac{1}{2}}, 1)$.*

Proof: By Lemma 6.6 the product of two functions in $\mathcal{M}_s \cap \mathcal{I}^{\rho,s}_{\gamma}$ is well-defined. One shows by dominated convergence that associativity holds and one shows similarly to the proof of Lemma 6.1 that $\mu_\gamma(f *_\gamma g)$ is well-defined and that it is equal to $\mu_\gamma(f)\mu_\gamma(g)$. By Lemma 5.10 $\mathcal{M}_s \cap \mathcal{I}^{\rho,s}_{\gamma}$ is commutative, if $s < 1$ and $u_\gamma \in \mathcal{M}_{q^{\frac{1}{2}}} \cap \mathcal{I}^{\rho,q^{\frac{1}{2}}}_{\gamma}$. $\square$

7 Convolution and Fourier transform

In this section I apply the results of Section 6 in order to extend Koornwinder's inversion results for the q-Fourier transform to be found in [Koo97]. At the end of the Section I shall also prove anaytically a result on the relation between q-convolution and q-Fourier transform that was proved in [KM94] in a different context (braided and with bosonic integral).
As it is stated in Section 2 the formal q-Fourier transform $\tilde{F}_\gamma$ is defined on $\mathcal{H}^{\mathrm{D}}\mathcal{I}_\gamma^{\mathcal{E}}$ and the q-Fourier transform is defined on $\mathcal{H}^{\mathrm{D}}\mathcal{I}_\gamma^{s\mathcal{E}}$. We have seen in Corollary 5.16 and Theorem 6.4 that

$$\mathcal{M}_{q^{\frac{1}{2}}} \cap \mathcal{I}_\gamma^\omega \simeq \mathcal{H}^{\mathrm{S}}\mathcal{I}_\gamma^\omega / [\mathcal{H}^{\mathrm{S}}\mathcal{I}_\gamma^\omega,\, \mathcal{H}^{\mathrm{S}}\mathcal{I}_\gamma^\omega]_* \simeq \mathcal{H}^{\mathrm{D}}\mathcal{I}_\gamma^\omega / [\mathcal{H}^{\mathrm{D}}\mathcal{I}_\gamma^\omega,\, \mathcal{H}^{\mathrm{D}}\mathcal{I}_\gamma^\omega]_*$$

and that

$$\mathcal{M}_{q^{\frac{1}{2}}} \cap \mathcal{I}_\gamma^{\mathcal{E}} \simeq \mathcal{H}^{\mathrm{S}}\mathcal{I}_\gamma^{\mathcal{E}} / [\mathcal{H}^{\mathrm{S}}\mathcal{I}_\gamma^{\mathcal{E}},\, \mathcal{H}^{\mathrm{S}}\mathcal{I}_\gamma^{\mathcal{E}}]_* \simeq \mathcal{H}^{\mathrm{D}}\mathcal{I}_\gamma^{\mathcal{E}} / [\mathcal{H}^{\mathrm{D}}\mathcal{I}_\gamma^{\mathcal{E}},\, \mathcal{H}^{\mathrm{D}}\mathcal{I}_\gamma^{\mathcal{E}}]_*$$

and that the kernel of the formal q-Fourier transform is exactly the commutator ideal. Hence it makes sense to look for an inverse of the formal q-Fourier transform $\tilde{\mathcal{F}}_\gamma$ on the image of $\mathcal{M}_{q^{\frac{1}{2}}} \cap \mathcal{I}_\gamma^\omega$ and $\mathcal{M}_{q^{\frac{1}{2}}} \cap \mathcal{I}_\gamma^{\mathcal{E}}$. Moreover, $\tilde{F}_\gamma(f)$ is also a well-defined function on a neighbourhood of 0 for $f \in \mathcal{M}_s \cap \mathcal{I}_\gamma^{\rho,s}$ and $s < q^{-\frac{1}{2}}$ and its radius of convergence will be $s^{-1}q^{-\frac{1}{2}} > 1$. $\tilde{F}_\gamma$ is injective on $\mathcal{M}_{q^{\frac{1}{2}}} \cap \mathcal{I}_\gamma^{\rho,q^{\frac{1}{2}}}$ by Lemma 5.10. Since $\tilde{\mathcal{F}}_\gamma$ is μ_γ up to a multiplicative shift of the variable, its inverse boils down to retrieving back a function knowing its q-moments. This can clearly be achieved by means of the functions $G_{r,\gamma}$'s as it was shown in Section 6.
Define the operator $\mathcal{G}_\gamma$ on the space of functions in $\mathcal{H}^{\mathrm{D}}$ whose radius of convergence is greater than q^{-1} as follows.

$$\mathcal{G}_\gamma(f) = \mathcal{G}_\gamma\Big(\sum_{k=0}^{\infty} c_k x^k\Big) = \sum_{k=0}^{\infty} i^k c_k (q;q)_k G_{k,\gamma} = \Big(\sum_{k=0}^{\infty} (-i)^k c_k (1-q)^k \partial^k\Big)\, u_\gamma \tag{7.1}$$

where $\sum_{k=0}^{\infty} c_k x^k$ is the power series expansion of f on a neighbourhood of 0. By definition of the $G_{k,\gamma}$'s it is clear that $c_q(\gamma)\mathcal{G}_\gamma$ is independent of γ.
Let us define the following spaces of functions: for $\alpha > 0$

$$\mathcal{E}_\alpha^\omega := \{f \in \mathcal{E} \mid f = \sum_k c_k x^k \text{ and } \exists b > 0 \,|, |c_k| = O(q^{\frac{1}{2}\alpha k^2} b^k) \text{ for } k \to \infty\}$$

i.e. $\mathcal{E}_\alpha^\omega$ is the space of functions of q-exponential growth of order α and finite type. Let

$$\mathcal{E}^\omega := \bigcup_{\alpha > 0} \mathcal{E}_\alpha^\omega$$

It is almost tautological that $\tilde{\mathcal{F}}_\gamma(\mathcal{I}_\gamma^{\mathcal{E}}) = \mathcal{E}$, $\tilde{\mathcal{F}}_\gamma(\mathcal{I}_\gamma^{\rho,s}) = \mathcal{H}^{\mathrm{D}}_{s^{-1}q^{-\frac{1}{2}}}$ where the lower index by $\mathcal{H}^{\mathrm{D}}$ will denote from now on the lower bound of the radius of convergence

(note that in [CK99] $\mathcal{H}^{\mathrm{D}}_a$ meant that the radius of convergence had to be greater or equal to a while here it denotes that the radius of convergence has to be *strictly greater* than a). We also have, $\tilde{\mathcal{F}}_\gamma(\mathcal{I}^\omega_{\gamma,\alpha}) = \mathcal{E}^\omega_\alpha$ and $\tilde{\mathcal{F}}_\gamma(\mathcal{I}^\omega_\gamma) = \mathcal{E}^\omega$. We get the following result:

Proposition 7.1 *$\tilde{\mathcal{F}}_\gamma$ defines an isomorphism of vector spaces between $\mathcal{M}_{q^{\frac{1}{2}}} \cap \mathcal{I}^\omega_{\gamma,\alpha}$ and $\mathcal{E}^\omega_\alpha$ and isomorphisms of algebras between $\mathcal{M}_{q^{\frac{1}{2}}} \cap \mathcal{I}^\omega_\gamma$ and $\mathcal{E}^\omega$, between $\mathcal{M}_{q^{\frac{1}{2}}} \cap \mathcal{I}^{\mathcal{E}}_\gamma$ and $\mathcal{E}$ and between $\mathcal{M}_s \cap \mathcal{I}^{\rho,s}_\gamma$ and $\mathcal{H}^{\mathrm{D}}_{s^{-1}q^{-\frac{1}{2}}}$ for $s \leq q^{\frac{1}{2}}$.*

Proof: By the discussion in the previous Sections one finds that $\mathcal{G}_\gamma(\mathcal{E}) = \mathcal{I}^{\mathcal{E}}_\gamma$, $\mathcal{G}_\gamma(\mathcal{E}^\omega) = \mathcal{I}^\omega_\gamma$ and $\mathcal{G}_\gamma(\mathcal{H}^{\mathrm{D}}_{s^{-1}q^{-\frac{1}{2}}}) = \mathcal{I}^{\rho,s}_\gamma$ if $s < 1$. By construction $\tilde{\mathcal{F}}_\gamma \circ \mathcal{G}_\gamma = \mathrm{id}$ on $\mathcal{H}^{\mathrm{D}}_{q^{-1}}$ and $\mathcal{G}_\gamma \circ \tilde{\mathcal{F}}_\gamma = \mathrm{id}$ on $\mathcal{I}^{\rho,s}_\gamma$ for $s \leq q^{\frac{1}{2}}$. The rest is clear. □

This inversion formula extends the inversion results in [Koo97]. Indeed Koornwinder showed therein that $\mathcal{F}_\gamma$ establishes a particular isomorphism between the vector space P_e of polynomials times $e_{q^2}(-X^2)$ and the vector space P_E of polynomials times $E_{q^2}(-q^2X^2)$ resembling the classical case. Since $P_e \subset \mathcal{H}^{\mathrm{S}}\mathcal{I}^{\mathrm{s}\omega}_\gamma$, by Proposition 2.3 $\tilde{\mathcal{F}}_\gamma = \mathcal{F}_\gamma$ on P_e and since $P_e \subset \mathcal{M}_{q^{\frac{1}{2}}} \cap \mathcal{I}^\omega_\gamma$, the two inverses must coincide on P_E. Koornwinder's inverse transform is essentially given by

$$(\mathcal{F}'_\gamma f)(y) := \frac{1}{c_q(\gamma)\, b_q} \int_{-1}^{1} e_q(ixy) f(x) d_q x \tag{7.2}$$

where $c_q(\gamma)$ is as in formula (2.6). Hence $(\mathcal{F}'_\gamma f)(y) \in \mathcal{H}^{\mathrm{S}}$ for every function f bounded in $(-1, 1)$. I will show that $\mathcal{F}'_\gamma$ and $\mathcal{G}_\gamma$ coincide on the whole $\mathcal{H}^{\mathrm{D}}_1$.

Let $f(x) = \sum_{k=0}^\infty c_k x^k$ for $|x| \leq q^{-1}$. For $|\mathrm{Im}(y)| < 1$, by dominated convergence we have

$$\frac{1}{c_q(\gamma)\, b_q} \int_\gamma e_q(ixy) f(x) d_q x = \frac{1}{c_q(\gamma)\, b_q} \sum_{k=0}^{\infty} c_k \int_{-1}^{1} e_q(ixy) x^k d_q x \tag{7.3}$$

If we denote q-differentiation with respect to y by ∂_y we have for $y \neq 0$

$$\partial_y \Big(\int_{-1}^{1} e_q(ixy) x^k d_q x \Big) = \frac{i}{(1-q)} \int_{-1}^{1} e_q(ixy) x^{k+1} d_q x \tag{7.4}$$

that can be extended by continuity at $y = 0$. Hence

$$(\mathcal{F}'_\gamma f)(y) = \frac{1}{c_q(\gamma)\, b_q} \sum_{k=0}^{\infty} (-i)^k c_k (1-q)^k \partial_y^k \int_{-1}^{1} e_q(ixy) d_q x \tag{7.5}$$

Besides

$$\begin{aligned}
&\int_{-1}^{1} e_q(ixy) d_q x = \frac{(1-q)}{(iy;q)_\infty} {}_2\phi_1(q, iy; 0; q, q) + \frac{(1-q)}{(-iy;q)_\infty} {}_2\phi_1(q, -iy; 0; q, q) \\
&= {}_2\phi_1(q, 0; q^2; q, iy) + {}_2\phi_1(q, 0; q^2; q, -iy) = 2\, {}_2\phi_1(0, 0; q^3; q^2; -y^2) \\
&= 2\, g_1 = 2\, c_q(\gamma) \frac{(q^2;q^2)_\infty}{(q^3;q^2)_\infty} u_\gamma = b_q c_q(\gamma) u_\gamma
\end{aligned}$$

where the second equality follows by (0.6.13) in [KS98]. Hence $\mathcal{G}_\gamma$ and $\mathcal{F}'_\gamma$ coincide on $\mathcal{H}^{\mathrm{D}}_{q^{-1}}$. As a byproduct we have found that

$$G_{k,\gamma} = \frac{(-i)^k}{(q;q)_k}\mathcal{F}'_q(x^k) = \frac{(-i)^k}{(q;q)_k}\frac{1}{b_q\, c_q(\gamma)}\int_{-1}^{1} x^k e_q(ixy) d_q x.$$

The $G_{k,\gamma}$'s are the basis corresponding to the basis of $\mathcal{E}$ given by monomials. There holds some sort of orthogonality between the two bases since $\int_\gamma G_{k,\gamma}x^l = q^{-\frac{1}{2}(k^2+k)}\delta_{k,l}$. This explains many of the properties of the $G_{k,\gamma}$'s with respect to product and integration. Observe also that as a consequence of formula (7.4) one can prove by induction that there are polynomials r_k and l_k of degree at most $k-1$ for which $y^{k+1}c_q(\gamma)G_{k,\gamma} = r_k(y)\cos_q(y) + l_k(y)\sin_q(x)$ where $\cos_q(y) := \frac{1}{2}(e_q(iy)+e_q(-iy))$ and $\sin_q(y) := \frac{1}{2i}(e_q(iy)-e_q(-iy))$. In some sense then expansion in terms of $G_{k,\gamma}$ is midway between a q-Fourier transform and a q-Fourier series. Interesting results about a q-analogue of Fourier series were obtained in [BS98], where continuous integrals are involved. It is interesting that the basic exponentials studied in [BS98] and references therein depend on two variables, and they are related to our g_m's for a particular value of the first variable. The connection between the two families can be the subject of future research.

Observe that $\mathcal{F}'_\gamma$ is a priori an inverse of $\tilde{\mathcal{F}}_\gamma$ and not of $\mathcal{F}_\gamma$ since $\mathcal{M}_{q^{\frac{1}{2}}} \cap \mathcal{I}^\omega_\gamma \not\subset \mathcal{I}^{\mathrm{s}\omega}_\gamma$. On the other hand, $\tilde{\mathcal{F}}_\gamma$ and $\mathcal{F}_\gamma$ coincide on the ideal I_e of $\mathcal{M}_{q^{\frac{1}{2}}} \cap \mathcal{I}^\omega_\gamma$ generated by $e_{q^2}(-X^2)$ because this ideal is contained in $\mathcal{H}^{\mathrm{S}}\mathcal{I}^{\mathrm{s}\omega}_\gamma$ by statement *2* of Proposition 2.2. By arguments similar to those in the proof of Proposition 5.3 in [CK99] one shows that the ideal I'_e generated by $e_{q^2}(-X^2)$ in $\mathcal{M}_{q^{\frac{1}{2}}} \cap \mathcal{I}^{\mathcal{E}}_\gamma$ is contained in $\mathcal{M}_{q^{\frac{1}{2}}} \cap \mathcal{I}^{s\mathcal{E}}_\gamma$. I_e can be described explicitely as the space of functions $F = f\, e_{q^2}(-X^2)$ where $f(x) = \sum_{k=0}^\infty c_k \tilde{h}_k(x;q)$ for which there are a c and an $\alpha > 0$ such that $|c_k| \le Cq^{\frac{1}{2}(\alpha+1)k^2}c^k$. This is shown using formula (6.2) in [CK99]. One shows that such an F has to be $F = g *_\gamma e_{q^2}(-X^2)$ for some $g \in \mathcal{H}^{\mathrm{D}}\mathcal{I}^\omega_{\gamma,\alpha}$, and using the results in the previous section one sees that g can be chosen to be in $\mathcal{M}_{q^{\frac{1}{2}}} \cap \mathcal{I}^\omega_{\gamma,\alpha}$. Similarly I'_e can be described explicitely as the space of functions of the form $f\, e_{q^2}(-X^2)$ where the coefficients c_k of the power series expansion of f are $O(q^{\frac{1}{2}(k)}R^k)$ for every R as $k \to \infty$. In the terminology of [Ram92], this means that f is q-Gevrey-Beurling of order -1.

We have:

$$\mathcal{F}_\gamma(I_e) = \tilde{\mathcal{F}}_\gamma((\mathcal{M}_{q^{\frac{1}{2}}} \cap \mathcal{I}^\omega_\gamma) *_\gamma e_{q^2}(-X^2)) = \mathcal{E}^\omega\, E_{q^2}(-q^2X^2) \tag{7.6}$$

and

$$\mathcal{F}_\gamma(I'_e) = \tilde{\mathcal{F}}_\gamma((\mathcal{M}_{q^{\frac{1}{2}}} \cap \mathcal{I}^{\mathcal{E}}_\gamma) *_\gamma e_{q^2}(-X^2)) = \mathcal{E}\, E_{q^2}(-q^2X^2) \tag{7.7}$$

by Proposition 2.3 and the results in [Koo97]. In particular, this proves that $\mathcal{F}'_\gamma(\mathcal{E}\, E_{q^2}(-q^2X^2)) \subseteq \mathcal{M}_{q^{\frac{1}{2}}} \cap \mathcal{I}^{s\mathcal{E}}_{\gamma,\frac{1}{2}}$ and $\mathcal{F}'_\gamma(\mathcal{E}^\omega\, E_{q^2}(-q^2X^2)) \subseteq \mathcal{M}_{q^{\frac{1}{2}}} \cap \mathcal{I}^{\mathrm{s}\omega}_{\gamma,\frac{1}{2}}$. On the

space $\mathcal{E}\,E_{q^2}(-q^2X^2)$ both $\mathcal{F}'_\gamma$ and $\mathcal{G}_\gamma$ coincide with the case $n=1$ and $\gamma=1$ of $F''(\mathrm{id},\gamma)$ in [Car99b] and [Car99a], with q^2 replaced by q. There, the integral is unbounded, but it coincides with a bounded one since $E_{q^2}(-q^2x^2)=0$ for $x=\pm q^{-k}$ with $k\in\mathbf{Z}_{\geq 1}$. The context of [Car99b] and [Car99a] was the braided setting (see also [KM94]) and the integral is also slightly different though. In our context, $F''(\mathrm{id},\gamma)$ for $\gamma=q$ is the operator

$$(\tilde{\mathcal{F}}'_q f)(y)=\sum_{e=0}^{\infty}\frac{i^e y^e}{(q;q)_e}\int_q fX^e$$

mapping $\mathcal{I}^{\omega}_{q,>1}$ to $\mathcal{E}$ and $\mathcal{I}^{\omega}_{q,1}$ to $\mathcal{H}^{\mathrm{D}}$.

Remark 7.2 If we view $\mathbf{C}[[x]]$ as the braided line $\mathcal{A}$ with the braided Hopf algebra structure recalled in Remark 3.2, then $F''(\mathrm{id},\gamma)$ is (up to a small change) the braided Fourier transform defined in [KM94] but with a nonbosonic integral. In this context the variable y lives in $\mathcal{B}$, another braided Hopf algebra. $\mathcal{B}$ is isomorphic, as an algebra, to $\mathbf{C}[[y]]$, it acts on $\mathcal{A}$ by letting y act on $f\in\mathcal{A}$ as ∂ and it has a nontrivial braided Hopf algebra pairing with $\mathcal{A}$, hence it is dual to $\mathcal{A}$.
The braided Hopf algebra structure on $\mathcal{B}$ is given by the braiding $\Psi(y^k\otimes y^l)=q^{kl}y^l\otimes y^k$, the comultiplication $\Delta(y)=y\otimes 1+1\otimes y$, the braided antipode $S(y^k)=(-1)^k q^{\binom{k}{2}}y^k$ and the counit $\varepsilon(y^k)=\delta_{k,0}$. Since the braiding between $\mathcal{A}$ and $\mathcal{B}$ is nontrivial, $1\otimes y$ and $x\otimes 1$ do not commute in $\mathcal{A}\otimes\mathcal{B}$. In particular $\psi(x^k\otimes\partial^l)=q^{-kl}\partial^l\otimes x^k$ and $(x\otimes\partial)^r=q^{-\binom{r}{2}}x^r\otimes\partial^r$. Hence, formally

$$\tilde{\mathcal{F}}_\gamma f=\Big(\int_\gamma\otimes\mathrm{id}\Big)(f\,E_q(ix\otimes y))\quad\text{and}\quad\tilde{\mathcal{F}}'_\gamma f=\Big(\int_\gamma\otimes SQ\Big)(f\,E_q(ix\otimes y))$$

where Q acts on $\mathcal{B}$ as q-shift of the indeterminate y. ♠

I conclude this section with a rigorous proof of a result in [KM94] concerning the behaviour of $\tilde{\mathcal{F}}'_\gamma$ with respect to the q-convolution.

Definition 7.3 *Let $\mathcal{C}[[x,\,y]]$ be the space of power series in x and y that converge in some polydisk $\{x\,|\,|x|<r_1\}\times\{y\,|\,|y|<r_2\}$. We define the operator*

$$\begin{array}{rcl}\Psi\colon\mathcal{C}[[x,\,y]] & \to & \mathcal{C}[[x,\,y]]\\ \displaystyle\sum_{n,m}a_{nm}x^n\,y^m & \mapsto & \displaystyle\sum_{n,m}a_{n,m}q^{nm}y^m\,x^n\end{array}$$

Remark 7.4 The operator Ψ is, in the braided context, the braiding from $\mathcal{A}\otimes\mathcal{A}$ or $\mathcal{B}\otimes\mathcal{B}$ to itself, see also Remark 3.2. ♠

Proposition 7.5 *Let $f \in \mathcal{H}^{\mathrm{D}}\mathcal{I}^{\omega}_{\gamma,\alpha}$ with $\alpha > 1$ and let $g \in \mathcal{H}^{\mathrm{D}}\mathcal{I}^{\omega}_{\gamma',\beta}$ with $\beta > 1$. If there holds $(\alpha-1)(\beta-1) > 1$ then*

$$\tilde{\mathcal{F}}'_{\gamma'}(f *_\gamma g) = m \circ \Psi^{-1}(\tilde{\mathcal{F}}_\gamma(f) \otimes \tilde{\mathcal{F}}'_{\gamma'}(g)) \tag{7.8}$$

and it converges absolutely everywhere. Equality (7.8) holds as equality of power series in $\mathcal{H}^{\mathrm{D}}$ if $(\alpha-1)(\beta-1) = 1$ or if f is as above and $g \in \mathcal{I}^{\mathrm{s}\omega}_{\gamma',1}$.

Proof: Observe that since $\mathcal{H}^{\mathrm{D}} \otimes \mathcal{H}^{\mathrm{D}}$ can be embedded in $\mathcal{C}[[x, y]]$, Ψ is well-defined on $\mathcal{H}^{\mathrm{D}} \otimes \mathcal{H}^{\mathrm{D}}$.
One sees that $\tilde{\mathcal{F}}_\gamma = Q \circ S \circ \tilde{\mathcal{F}}'_\gamma$, with S as in Definition 3.1. Observe that $S \circ m = m \circ (S \otimes S) \circ \Psi$ where m is the ordinary product, Q commutes with Ψ and S on power series and $Q \circ m = m \circ (Q \otimes Q)$. By Proposition 2.3

$$\begin{aligned}
&S\tilde{\mathcal{F}}'_{\gamma'}(f *_\gamma g)\\
&= m(S \otimes S)(\tilde{\mathcal{F}}'_\gamma(f) \otimes \tilde{\mathcal{F}}'_\gamma(g)) = S \circ m \circ \Psi^{-1}(\tilde{\mathcal{F}}'_\gamma(f) \otimes \tilde{\mathcal{F}}'_{\gamma'}(g))
\end{aligned}$$

where the composite $S \circ m \circ \Psi^{-1}$ is a well-defined operator on $\mathcal{C}[[x, y]]$ although ψ^{-1} itself may not.
If $(\alpha-1)(\beta-1) > 1$ by Proposition 4.6 in [CK99] $\tilde{\mathcal{F}}'_{\gamma'}(f *_\gamma g) \in \mathcal{E}$. In this case we can multiplyboth sides by the inverse of the antipode S without problems, obtaining equality (7.8). Both sides of the equality will be absolutely convergent everywhere. If $(\alpha-1)(\beta-1) = 1$ or if $\alpha > 1$ and $g \in \mathcal{I}^{\mathrm{s}\omega}_{\gamma',1}$, then $\tilde{\mathcal{F}}'_{\gamma'}(f *_\gamma g)$ converges absolutely on a neighbourhood of zero by statement *3* of Proposition 2.2 and we get the statement. □

Remark 7.6 One could also check that the series $m\Psi^{-1}(\tilde{\mathcal{F}}'_\gamma(f) \otimes \tilde{\mathcal{F}}'_{\gamma'}(g))$ converges in a neighbourhood of zero by direct computation using the fact that if the power series $F(x) = \sum_{n=0}^{\infty} a_n x^n$ is such that $|a_n| \leq C a^n q^{\frac{(\alpha-1)}{2}n^2}$ for some C, $a \geq 0$ and some $\alpha > 1$ and if the power series $G(x) = \sum_{n=0}^{\infty} b_n x^n$ is such that $|b_n| \leq B b^n q^{\frac{(\beta-1)}{2}n^2}$ for some B, $b \geq 0$ and some $\beta > 1$ and if $(\alpha-1)(\beta-1) > 1$ then

$$m \circ \Psi^{-1}(F(x) \otimes G(x)) = \sum_{t=0}^{\infty} \Big[\sum_{m+n=t} a_m b_n q^{-nm} \Big] x^t \quad \text{and}$$

$$\begin{aligned}
&\Big| \sum_{m+n=t} a_m b_n q^{-nm} \Big|\\
&\leq CB \sum_{m+n=t} a^m b^n q^{\frac{1}{2}((\alpha-1)-\frac{1}{(\beta-1)})m^2} q^{\frac{1}{2(\beta-1)}(m^2+(\beta-1)^2n^2-2(\beta-1)nm)}\\
&\leq D(\max(a, b))^t \sum_{m=0}^{t} q^{\frac{1}{2}((\alpha-1)-\frac{1}{(\beta-1)})m^2}
\end{aligned}$$

♠

Remark 7.7 In [Zha99] , where $q > 1$, an analytic inversion formula along a direction in $\mathbf{C}$ for the q^{-1}-Borel transform is given for functions with a particular growth. Recall that on $\mathcal{H}^{\mathrm{D}}$, S is essentially the q-Borel transform. See also [Ram92] for the isomorphism of vector spaces between $\mathcal{H}^{\mathrm{D}}$ and q-Gevrey series. ♠

8 Invertibility of functions

Next question is whether given $f \in \mathcal{H}^{\mathrm{D}}\mathcal{I}_\gamma^{\mathcal{E}}$ there exists a function for which $f *_\gamma g = u_\gamma$ and in that case whether $g *_\gamma f$ is also well-defined and equal to u_γ. If a left inverse exists, it will not be unique since any element in $g + [\mathcal{H}^{\mathrm{D}}\mathcal{I}_\gamma^{\mathcal{E}}, \mathcal{H}^{\mathrm{D}}\mathcal{I}_\gamma^{\mathcal{E}}]_*$ will also be a left inverse.
By formula (2.5) it follows that a necessary condition for invertibility of a function of left type is that $\mu_{0,\gamma}(f) = \int_\gamma f \neq 0$. In particular, odd functions can never be invertible.
Observe also that if an inverse of $\mu_\gamma(f)$ exists, then it might not correspond to a function of $\mathcal{I}_\gamma^{\mathcal{E}}$, since the only functions that are invertible in $\mathcal{E}$ are those with no zeroes.
On the other hand if $\mu_\gamma(f)(t) \neq 0$ for $|t| < \rho$ then the inverse of $\mu_\gamma(f)$ will be defined and analytic for $|t| < \rho$ and its power series expansion on this disk will be

$$\nu(t) = \sum_{k=0}^{\infty} \frac{d_k t^k}{[k]_q!} \quad \text{and } |d_k| = O(\sigma^k) \text{ for } k \to \infty \text{ for every } \sigma > \rho^{-1}(1-q)^{-1}. \tag{8.1}$$

In Section 6 we showed how to construct a function $g \in \mathcal{I}_\gamma^\infty$ such that $\mu_\gamma(g)(t) = \nu(t)$ on a neighbourhood of 0 at least if ρ is big enough.
Observe that if $f \in \mathcal{I}_\gamma^{\mathcal{E}}$ then for any $p \in \mathbf{Z}$ we have $Q^{-p}f \in \mathcal{I}_\gamma^{\mathcal{E}}$ and $\mu_\gamma(Q^{-p}f)(t) = q^p\mu_\gamma(f)(q^p t)$, because $\int_\gamma Q(F) = q^{-1}\int_\gamma F$ for every $F \in \mathcal{I}_\gamma$. Hence if $\mu_\gamma(f)(t) \neq 0$ for $|t| < \rho$, then $\mu_\gamma(Q^{-p}f)(t) \neq 0$ for $|t| < \rho q^{-p}$. This tells that the conditions on f are satisfied at least for its (big) q-shifts.

Proposition 8.1 *Let $f \in \mathcal{I}_\gamma^\omega$ and let $\mu_\gamma(f)(t) \neq 0$ for $|t| < \rho$. If $\rho > (1-q)^{-1}$ there exists a function $g \in \mathcal{M}_s$ with $s < q^{-\frac{1}{2}}$ such that $\mu_\gamma(g)\mu_\gamma(f) = 1$. If moreover, $\rho > q^{-1}(1-q)^{-1}$ then $g \in \mathcal{M}_{q^{\frac{1}{2}}} \cap \mathcal{I}_\gamma^{\rho,q^{-1}}$. In this case, $f *_\gamma g = u_\gamma$.*

Proof: It is a consequence of Lemma's 6.1, 6.2 and 6.6. The fact that $f *_\gamma g = u_\gamma$ follows from Lemma 5.10. □

Corollary 8.2 *Let f, g, ρ be defined as in Proposition 8.1 with $\rho > (1-q)^{-1}q^{-1}$ and let $f \in \mathcal{M}_{q^{\frac{1}{2}}}$. Then $f *_\gamma g = g *_\gamma f = u_\gamma$.*

Proof: We have to show that $g *_\gamma f = u_\gamma$. By Lemma 6.6 $g *_\gamma f$ is a well-defined function in $\mathcal{M}_{q^{\frac{1}{2}}}$, hence $g *_\gamma f \in \mathcal{I}_\gamma^\infty$. Formula (2.5) holds by Corollary 6.7, so that by Lemma 5.10 the statement follows. $\square$

Proposition 8.3 *Let $g \in \mathcal{I}_\gamma^\infty$ be such that $\mu_\gamma(g) = \sum_{k=0}^\infty \frac{d_k}{[k]_q!} t^k$ with $|d_k| = O(\sigma^k)$ for $k \to \infty$, and let h be a function defined, together with its q-derivatives, on a domain Ω. If there is an $M \in (0, \sigma^{-1}(1-q)^{-1})$ for which $|\partial^k h(x)| = O(M^k)$ for $x \in \Omega$, then $g *_\gamma h$ is well-defined on Ω. In particular, if $h \in \mathcal{H}_R^{\mathrm{D}}$ for $R > \sigma$ then $g *_\gamma h \in \mathcal{H}_R^{\mathrm{D}}$, and if $h \in \mathcal{E}$, then $g *_\gamma h \in \mathcal{E}$.*

Proof: The proof follows like the proof of Lemma 3.4 in [CK99]. $\square$

Corollary 8.4 *Let f, g and ρ be defined as in Proposition 8.1, with $\rho > (1-q)^{-1}q^{-1}$. Let $h \in \mathcal{H}_R^{\mathrm{D}}$ for some $R > \rho^{-1}(1-q)^{-1}$. Then $g *_\gamma h \in \mathcal{H}_R^{\mathrm{D}}$. In particular, if $h \in \mathcal{E}$ then $g *_\gamma h \in \mathcal{E}$.*

Proof: Clear by the previous results. $\square$

9 Convolution and q-differential equations

In this section I will show a link between invertibility of a function in $\mathcal{M}_{q^{\frac{1}{2}}} \cap \mathcal{I}_\gamma^{\mathcal{E}}$ and the solution of a q-differential equation with constant coefficients on a fixed q-lattice $L(\gamma)$. A *q-differential equation* is an equation of the form $LY = F$ where F is a given function , Y is an unknown and $L \in \mathbf{C}[\partial] \subset \mathbf{C}[x^{-1}, Q]$. Hence such an equation can be reduced to a q-difference equation with polynomial coefficients and with leading term $= 1$ by multiplying both sides by $(-1)^n c_n^{-1}(1-q)^n q^{\binom{n}{2}} x^n$ where n is the order of the equation and c_n is the coefficient of ∂^n in L. The associated q-difference equation are always regular singular at 0 (see [Ram92] for a definition) and their characteristic equation (see [Ada31]) has roots $1, q, \ldots, q^{n-1}$. Many interesting results about the behaviour of the solutions were known already in the 30's, see [Ada31] and [Ada29], where in particular the solutions for homogeneous equations are described. Many methods for solving q-difference equations are known, hence I do not think that using q-convolution shall simplify the problem in general. It offers though a different interpretation and can help to describe the type of solutions one could find and to find a particular solution in special cases, as I shall show.

Let

$$LY = \sum_{n=0}^{N} c_k \partial^k Y = F \tag{9.1}$$

for a given function F and an unknown Y. Equation (9.1) is equivalent to

$$\sum_{n=0}^{N} c_k \partial^k u_\gamma *_\gamma Y = \left(\sum_{k=0}^{N} (-1)^k c_k [k]_q! G_{k,\gamma} \right) *_\gamma Y = D_L *_\gamma Y = F. \tag{9.2}$$

where D_L is of left type for every $\alpha > 0$ since $\mu_\gamma(D)$ is a polynomial.
If $c_0 \neq 0$ we can invert $\mu_\gamma(D_L) = \sum_{n=0}^{N}(-1)^n c_n t^n$ on a neighbourhood of 0.
If $c_0 = c_1 = \cdots = c_{l-1} = 0$ and $c_l \neq 0$, then equation (9.1) becomes

$$\sum_{p=0}^{N-l} c_{p+l}\partial^p(\partial^l Y) = F. \tag{9.3}$$

If we can solve $\sum_{p=0}^{N-p} c_{p+l}\partial^p(Z) = F$, and if the solution behaves well (for instance, if it belongs to $\mathcal{H}^{\mathrm{D}}$), then we can determine solutions of equation (9.3) applying l times indefinite q-integration $\int_0^x f(t)d_qt$. As in classical integration, q-integration determines q-primitives up to a constant.
Now suppose $c_0 \neq 0$. $\mu_\gamma(D_L)$ is a polynomial, hence it will have zeroes. If $\mu_\gamma(D_L)(t) \neq 0$ for $|t| < \rho$ and $\rho > (1-q)^{-1}$, then we can construct g, a left inverse of $D_L \in \mathcal{M}_{q^{\frac{1}{2}}} \cap \mathcal{I}_\gamma^\omega$ and it will belong to $\mathcal{M}_s$ for some $s < q^{-\frac{1}{2}}$. The function g should be seen as a q-analogue of the classical fundamental solution associated to the differential equation $Ly = F$, since for a fundamental solution there should hold: $Lg = \delta$ where δ is Dirac's delta. By the results in the preceding sections, if ρ is big enough we may compute $g *_\gamma F$. Even if the function g does not belong to $\mathcal{I}_\gamma^\infty$, we still can formally compute $g *_\gamma F$ using the coefficients of its expansion with respect to the $G_{k,\gamma}$'s as if they were really q-moments. Then

- If $F \in \mathcal{M}_s$ with $s < q^{-\frac{1}{2}}$, then the formal product $g *_\gamma F \in \mathcal{M}_r$ for some $r < q^{-\frac{1}{2}}$.
- If $F \in \mathcal{M}_s$ for $s < q^{-\frac{1}{2}}$ and $\rho > (1-q)^{-1}s^{-1}q^{-\frac{1}{2}}$ then $g *_\gamma F \in \mathcal{M}_s$.
- If $F \in \mathcal{H}^{\mathrm{D}}_{\rho'}$ for $\rho' > \rho^{-1}(1-q)^{-1}$, then $g \in \mathcal{I}_\gamma^\infty$ and $g *_\gamma F \in \mathcal{H}^{\mathrm{D}}_{\rho'}$.
- If $F \in \mathcal{E}$ then $g *_\gamma F \in \mathcal{E}$.

In all those cases, both $D *_\gamma (g *_\gamma F)$ and $(D *_\gamma g) *_\gamma F$ are well-defined, hence they are both equal to F by dominated convergence. This implies that $g *_\gamma F$ is a solution of equation (9.1). If we replaced g by another function g' for which $\mu_\gamma(g')\mu_\gamma(f) = 1$ on a neighbourhood of zero, then $g *_\gamma h = g' *_\gamma h$ for every h for which the product is defined, hence we would get the same solution. Moreover, unicity of a possible solution in $\mathcal{M}_s$ for $s < 1$ follows by determinacy of the q-moment problem on $\mathcal{M}_s$. If F is a polynomial instead, the solution obtained by this construction will be again a polynomial.
Suppose now that $c_0 \neq 0$ but $\rho \leq (1-q)^{-1}q^{-1}$. We can find a $p \in \mathbf{Z}_{\geq 0}$ for which $\rho_p := q^{-p}\rho > (1-q)^{-1}q^{-1}$. Observe that $q^{-p}\rho$ is related to the homogeneous equation

$$\Big(\sum_{n=0}^{N} c_n q^{pn}\partial^n\Big)Y = 0 \tag{9.4}$$

as ρ is related to equation (9.1). In this case, the role of D is played by the function $D_p := \sum_{k=0}^{N}(-1)^k c_k [k]_q! q^{pk} G_{k,\gamma} \in \mathcal{M}_{q^{\frac{1}{2}}} \cap \mathcal{I}^{\omega}_{\gamma}$. The inverse of $\mu_\gamma(D_p)(t) = \mu_\gamma(D)(q^p t)$ is analytic for $|t| < q^{-p}\rho = \rho_p$.
The corresponding function g_p is well-defined and it is such that $D_p *_\gamma g_p = u_\gamma$ by Corollary 8.4. If $F \in \mathcal{H}^{\mathrm{D}}_{\rho'}$ with $\rho' > \rho^{-1}(1-q)^{-1}$, then the function $F_p := Q^{-p}F \in \mathcal{H}^{\mathrm{D}}_{\rho'_p}$ where $\rho'_p = \rho q^p > q^p \rho^{-1}(1-q)^{-1} = \rho_p^{-1}(1-q)^{-1}$. Hence $Y_p := g_p *_\gamma F_p \in \mathcal{H}^{\mathrm{D}}_{\rho'_p}$ and it is a solution of:

$$\sum_{n=0}^{N} c_n q^{pn} \partial^n Y = F_p. \tag{9.5}$$

Then $Q^p Y_p \in \mathcal{H}^{\mathrm{D}}_{\rho'}$ and

$$\sum_{n=0}^{N} c_n \partial^n Q^p Y_p = \sum_{n=0}^{N} c_n q^{np} Q^p \partial^n Y_p = Q^p F_p = F \tag{9.6}$$

so that $Q^p Y_p$ is a solution of equation (9.1). This shows that if ρ' is big enough, the inverse of D may not be well-defined but we could still apply this method in order to find a particular solution.

Example 9.1 Consider the equation

$$Y - q^{2r}\partial^2 Y = e_{q^2}(-x^2). \tag{9.7}$$

This equation is equivalent to $D *_\gamma Y = e_{q^2}(-x^2)$ with $D = u_\gamma - q^{2r}[2]_q! G_{2,\gamma}$. $\mu_\gamma(D)(t) = 1 - q^{2r}t^2$, so that its inverse for $|t| < q^{-2r}$ is equal to $\nu(t) = \sum_{p=0}^{\infty}(q^{2r}t^2)^p = \sum_{p=0}^{\infty} q^{2rp}t^{2p}$. For r big enough, $\rho = q^{-r} > (1-q)^{-1}$, hence

$$g := \sum_{p=0}^{\infty}[2p]_q! q^{2rp} G_{2p,\gamma} = \sum_{p=0}^{\infty} q^{2rp}\partial^{2p} u_\gamma$$

is well-defined, as well as

$$g *_\gamma e_{q^2}(-x^2) = \sum_{p=0}^{\infty} q^{2rp}\partial^{2p} e_{q^2}(-x^2) = \sum_{p=0}^{\infty} \frac{q^{2rp}q^{2p^2-p}}{(1-q)^{2p}} \tilde{h}_{2p}(x;q) e_{q^2}(-x^2)$$

which is a solution of the equation. ♠

If $F \in \mathcal{M}_s$ with $s < 1$ then the method of q-shift does not apply since $\rho' = 1$ can never be greater than $\rho^{-1}(1-q)^{-1} > 1$. In this case, one can approximate the solution by multiplying F by finite linear combinations g_n of $G_{k,\gamma}$'s converging to g in the topology described in the previous Section. Then since the q-convolution product is continuous with respect to this topology, $g_n *_\gamma F$ will converge to Y with respect to this topology.

Remark 9.2 The approach we used to solve a q-differential equation can be translated into applying $\tilde{\mathcal{F}}_\gamma$ to both sides of equation (9.1) and considering whether $\mathcal{G}_\gamma$ can be applied to $(\tilde{\mathcal{F}}_\gamma(D))^{-1}\tilde{\mathcal{F}}_\gamma(F)$. ♠

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References

[Ada25] C. R. Adams. Note on the existence of analytic solutions of non homogeneous linear q-difference equations, ordinary and partial. *Annals of Mathematics*, 25:73–83, 1925.

[Ada29] C. R. Adams. On the linear ordinary q-difference equation. *Annals of Mathematics*, 30:195–205, 1929.

[Ada31] C. R. Adams. Linear q-Difference Equations. *Bull. AMS*, 31:361–382, 1931.

[Béz93] J. P. Bézivin. Sur les équations fonctionelles aux q-differences. *Aequationes Mathematicae*, 43:159–176, 1993.

[BS98] J. Bustoz and S. K. Suslov. Basic analog of Fourier series on a q-quadratic grid. *Methods and Applications of Analysis*, 5:1–38, 1998.

[Car99a] G. Carnovale. *Algebraic and Analytic Aspects of the Quantum Yang-Baxter equation.* 1999. PhD Dissertation, University of Utrecht.

[Car99b] G. Carnovale. On the braided Fourier transform in the n-dimensional quantum space. *J. Math. Phys.*, 40(11):5972–5997, 1999.

[CK99] G. Carnovale and T. Koornwinder. A q-analogue of convolution on the line. Technical report, 1999. math. CA/9909025.

[GR90] G. Gasper and M. Rahman. *Basic hypergeometric series.* Cambridge University Press, Cambridge, 1990.

[KM94] A. Kempf and S. Majid. Algebraic q-integration and Fourier theory on quantum and braided spaces. *J. Math. Phys.*, 35(12):6802–6837, 1994.

[Koo90] T. Koornwinder. Orthogonal polynomials in connection with quantum groups. In P. Nevai, editor, *Proceedings NATO ASI on Orthogonal polynomials, Columbus, Ohio, 1989*, pages 257–292. Kluwer Academic Press, 1990.

[Koo97] T. Koornwinder. Special Functions and q-commuting variables. In M.E.H Ismail, D.R. Masson, and M. Rahman, editors, *Special Functions, q-series and related topics - Fields Institute Communications 14*, pages 131–166. AMS, 1997.

[Koo99] T. Koornwinder. Some simple applications and variants of the q-binomial formula. 1999. informal paper.

[KS92] T. Koornwinder and R. F. Swarttouw. On q-analogues of the Fourier and Hankel transforms. *Trans. Am. Math. Soc.*, 333:445–461, 1992.

[KS98] R. Koekoek and R. Swarttouw. The Askey-scheme of hypergeometric orthogonal polynomials and its q-analogue. Technical report, Delft University of Technology, Faculty TWI, 1998. Report 98-17.

[Maj93] S. Majid. Braided momentum in the q-Poincaré group. *J. Math. Phys.*, 34:2045–2058, 1993.

[Maj95] S. Majid. *Foundations of quantum groups*. Cambridge University Press, 1995.

[OR97] M. Olshanetsky and V. Rogov. The q-Fourier transform of q-distributions. Technical report, IHES, December 1997. QA/9712055.

[Ram92] J. P. Ramis. About the growth of entire functions solutions of linear algebraic q-difference equations. *Ann. de la Fac. de Toulouse, Série 6*, I:53–94, 1992.

[Ryd21] F. Ryde. *A contribution to the theory of linear homogeneous geometric difference equations (q-difference equations).* 1921. PhD Dissertation, Lund.

[Zha99] C. Zhang. Développements asymptotiques qGevrey et séries Gq-sommables. *Ann. Inst. Fourier*, 49:227–261, 1999.